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Central Bank Digital Currency and Transmission of Monetary Policy

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Abstract

How does the transmission of monetary policy change when a central bank digital currency (CBDC) is introduced in the economy? Do aspects of CBDC design, such as how substitutable it is with bank deposits and whether it is interest bearing, matter? We study these questions in a general equilibrium model with nominal rigidities, liquidity frictions, and a banking sector where commercial banks face a leverage constraint. In the model, CBDC and commercial bank deposits can be used as a means of payments, and they provide liquidity services to households. Banks issue deposits and extend loans to firms, and bank deposits are backed by loans and central bank reserves. We find that the effects of a canonical monetary policy shock, a shock to the Taylor rule that governs interest on central bank reserves, is magnified with the introduction of a fixed-interest-rate CBDC. More generally, whether CBDC magnifies or abates the response of the economy depends on the type of shock (e.g., interest rate or quantity of reserves shock). We also find that the response of the economy depends on the monetary policy framework—whether the central bank implements monetary policy through reserves or through CBDC—as well as central bank balance sheet rules that govern the quantity of CBDC and reserves.

Topics: Digital currencies and fintech; Monetary policy; Monetary policy framework; Monetary policy transmission; Interest rates

JEL codes: E31, E4, E50, E58, G21, G51

Résumé

Comment l'introduction d'une monnaie numérique de banque centrale (MNBC) dans l'économie influence-t-elle la transmission de la politique monétaire? Les caractéristiques de cette MNBC, notamment à quel point elle est substituable aux dépôts bancaires ou si elle porte intérêt, ont-elles de l'importance? Nous analysons ces questions au moyen d'un modèle d'équilibre général comportant des rigidités nominales, des frictions relatives à la liquidité et un secteur bancaire dans lequel les banques commerciales sont assujetties à une contrainte de levier d'endettement. Dans ce modèle, la MNBC et les dépôts dans les banques commerciales peuvent être utilisés comme modes de paiement et constituent un vecteur de liquidité pour les ménages. Les banques offrent des services de dépôt et octroient des prêts aux entreprises, et les dépôts bancaires sont garantis par des prêts et les réserves de la banque centrale. Nous constatons que les effets d'un choc de politique monétaire standard – c'est-à-dire un choc touchant le taux d'intérêt des réserves de la banque centrale, régi par la règle de Taylor – sont amplifiés par l'introduction d'une MNBC à taux d'intérêt fixe. De façon plus générale, c'est le type de choc (p. ex., choc au niveau des taux d'intérêt ou de la quantité de réserves) qui détermine si la MNBC amplifie ou atténue la réaction de l'économie. Nous notons aussi que cette réaction dépend du cadre de politique monétaire, soit le moyen utilisé par la banque centrale pour mettre en œuvre la politique monétaire (les réserves ou la MNBC), et des règles qui encadrent le bilan de la banque centrale et régissent la quantité de MNBC et de réserves.

Sujets : Monnaies numériques et technologies financières; Politique monétaire; Cadre de la politique monétaire; Transmission de la politique monétaire; Taux d'intérêt

Codes JEL : E31, E4, E50, E58, G21, G51

1 Introduction

Many central banks are contemplating issuing a central bank digital currency (CBDC) and are concerned about the implications. As a means of payment, a CBDC would compete with bank deposits and thereby have implications for the banking and financial system. As a store of value, CBDC would be used along with bank deposits, government bonds, and other safe assets, which would have macroeconomics implications. A growing body of literature has studied such implications recently. However, most studies have focused on the steady-state or long-run effects. Fewer papers have studied the transitory and short-run effects of shocks on both banking and the macroeconomy in the presence of CBDC.[1]

In this paper, we propose a framework that can help understand the effects of shocks on output, consumption, inflation, and investment through various channels in the presence of a CBDC. The framework is built on a New Keynesian (NK) model with financial frictions that includes two new key features. Firstly, CBDC competes with deposits in providing liquidity services to households,[2] with the elasticity of substitution between the CBDC and deposits a design feature of the CBDC. Secondly, the model considers a general equilibrium channel in that, on the one hand, the production of final goods relies on loans made by banks. On the other hand, consumers use deposits issued by banks as a source of liquidity needed to buy the final goods. As the demand or supply of CBDC changes, so does the demand for bank deposits. This changes banks' cost of funding, which in turn alters the cost of loans for firms. Subsequently, the supply side of the economy is affected. Overall, our framework offers a flexible approach to studying the impact of CBDC on monetary policy transmission by considering its effects on both the demand and supply sides of the economy.

To understand the basic economic forces in the model, we first study the impact of a canonical monetary policy shock, an increase in the interest rate on central bank reserves, in the absence of a CBDC. There are three main channels through which the shock transmits to the economy. Firstly, a standard NK channel, where an increase in bond interest rates leads to lower current consumption and thus reduced aggregate demand and output, which impacts inflation via the NK Phillips curve. Secondly, a New Monetarist (NM) channel, where a narrower spread between illiquid bond and bank deposit rates reduces the opportunity cost of holding deposits (cost of liquidity), thereby boosting deposit demand, consumption, labor supply, and ultimately output. This offsets some effects of the initial shock. Lastly, a supply channel, where higher capital costs arising from higher costs of bank funding reduces investment, leading to a decline in output. These three channels together illustrate the interplay of economic forces following a monetary policy shock, showing both contractionary and expansionary effects on the economy.

[1]For steady-state effects, see for example, Barrdear and Kumhof (2022); Davoodalhosseini (2022); and Chiu and Davoodalhosseini (2023). For transitory effects, see for example, Minesso et al. (2022), who examine how a CDBC affects the international transmission of monetary policy and technology shocks.

[2]We extend the model in the appendix to include cash. In the main text, we do not include cash to make the exposition simpler.

We then turn to studying the effects of shocks in the presence of a CBDC. Given that many central banks are considering a zero-interest CBDC, we focus on such a CBDC in this exercise. The first takeaway from our paper is that the introduction of a zero-interest CBDC magnifies the effects of a traditional monetary policy shock. This amplification remains regardless of the substitutability between deposits and the CBDC. An increase in the reserves interest rate means that the leverage constraint becomes less costly for banks. Banks are therefore willing to offer more deposits. To attract depositors to hold more deposits, the opportunity cost of holding deposits, i.e., the spread between illiquid bonds and bank deposits (deposit spread), should fall. At the same time, the illiquid bonds interest rate tends to rise in response to the increase in the reserves rate. In the absence of a CBDC, the decline in the deposit spread suggests that the effective real wage increases, so there will be a subsequent boost in labor supply, consumption, and output. This is the same NM channel stated above. However, when a zero-interest CBDC is introduced, a fixed interest rate on the CBDC implies that the rise in the illiquid bond rate elevates the opportunity cost of holding CBDC. Consequently, the presence of a CBDC tends to heighten the overall cost of liquidity in the economy, thereby attenuating the NM channel. As the NM channel counteracts the NK channel, its attenuation means amplification of the decrease in output.

Whether a CBDC amplifies or abates the effects of a shock compared with the benchmark model without a CBDC also depends on the type of monetary policy shock. In contrast to the response to a standard monetary policy shock that is a shock to the interest on reserves, we find that a CBDC *abates* the effects on consumption, output, and inflation of a *reserve quantity* shock. A positive reserve quantity shock is expansionary with or without a CBDC. Without a CBDC, investment declines considerably initially and starts recovering only after a few quarters. With a CBDC, the decline in investment is much smaller. This explains why the overall effect on output and consequently consumption is dampened with a CBDC.

Another takeaway from our paper is that the response of the economy depends on the monetary policy framework, whether the central bank implements monetary policy through reserves or through a CBDC. To see this point, we compare the outcomes in response to the same shock with two monetary policy rules: one in which the central bank uses a Taylor rule to set the *CBDC interest rate* and another one in which it uses a Taylor rule to set the *reserves interest rate*. Our analysis focuses on the responses to a positive CBDC interest rate shock under these two monetary policy rules. We find that when CBDC serves as the primary tool for monetary policy, an increase in the CBDC interest rate is contractionary, resembling traditional monetary policy effects observed in most NK models. Alternatively, when reserves are the main tool, the shock to the CBDC interest rate is expansionary due to the reduced opportunity costs of liquidity for households, leading to increased labor supply and consumption—a mechanism akin to the NM channel.[3] This result underscores the importance of monetary policy framework in shaping economic responses.

Finally, we show that the response of the economy to a standard monetary policy shock depends on balance sheet quantity rules that govern the evolution of the central bank balance

[3]Here, the CBDC interest rate changes exogenously and does not follow a Taylor rule.

sheet variables. To best illustrate this point, we compare two scenarios, one in which the central bank fixes the CBDC rate and one in which it fixes the CBDC quantity. In response to a standard monetary policy shock (i.e., a shock in the Taylor rule that governs the interest rate on reserves), fixing the CBDC interest rate leads to a significant fall in output and consumption compared to fixing the CBDC quantity. When the CBDC interest rate is fixed, the opportunity cost of holding CBDC rises with an increase in the illiquid bond rate, making liquidity more expensive and reducing consumption and output further. When the CBDC quantity is fixed and its interest rate is flexible, the CBDC rate increases in response to pressure from rising deposit rates, activating the NM channel and mitigating contractionary effects of the shock. This exercise thus highlights the importance of the balance sheet quantity rules, which is in stark contrast to standard NK models where the quantity of the real balance has no independent effect on real variables once the short-term nominal rate is determined.

Literature. The literature on CBDCs has grown significantly in recent years, covering a wide range of topics. First, we compare our results with some closely related papers including Piazzesi et al. (2019), regarding the transmission of monetary policy shocks, and Chiu et al. (2023), regarding the main economic forces at play in the steady state. Next, we discuss the related literature more broadly.

Piazzesi et al. (2019) study an NK model with money in the utility function and complementarity between consumption and money. There are two key differences between our model and theirs. First, in our model, the CBDC and bank deposits *both* provide liquidity to households and could be substitutes or complements depending on the design of the CBDC. In their paper, agents either use central bank money (Section 2) or bank deposits (Section 3) and it does not consider the effects of a CBDC on the banking system. Second, banks in our model lend to firms to finance their capital expenditure. In Piazzesi et al. (2019), banks simply invest in some assets with an exogenous rate of return. Modeling the interactions between deposits and other means of payments provides insight into how changes in the interest rate of a CBDC or its design features affect demand for bank deposits (the first difference), which in turn changes the cost of funding for firms, eventually affecting the supply side of the economy (the second difference).

In Chiu and Davoodalhosseini (2023), banks issue deposits and extend loans to firms, and the CBDC and bank deposits compete in the sense that both can be used in a fraction of transactions. The similarity between their model and ours is that in both models, the supply side of the economy is affected by the demand for means of payments. They find that an increase in the CBDC interest rate can improve intermediation, consumption, and output in the steady state because an increase in the CBDC rate increases aggregate demand, leading firms to demand more loans to finance production. Thus, intermediation, consumption, and output can increase. A similar channel works in our model. However, Chiu and Davoodalhosseini (2023) do not study dynamic responses to shocks and focus only on the steady-state analysis. Moreover, our model allows for a wider range of design features in terms of complementarity or substitutability between a CBDC and other payment methods. Finally, in our model, banks face financial frictions, which gives rise to demand for central bank reserves. As a result, monetary policy affects this economy via a richer set of policy tools (i.e., inflation rate, the CBDC interest rate and quantity, and the reserves rate and quantity) relative to their paper, in which there is no demand for central bank reserves and the only central bank policies are the inflation rate and the interest rate on the CBDC.

Abad et al. (2023) study the implications of a CBDC for macroeconomic variables, focusing on frictions in the interbank market. Similar to our paper, they have a money-in-the-utility function framework, but they do not consider the complementarity between consumption and money that activates the NM channel in our paper. Moroever, most of their analysis focuses on the long-term implications of a CBDC and only a smaller part addresses transitional dynamics to an economy with a CBDC. They do not study effects of different shocks.

Paul et al. (2024) study the long-term welfare implications of introducing a CBDC as well as the transitional dynamics following the introduction of a CBDC in a closed economy framework. Again, they do not incorporate the NM channel. They also find that the implementation of a CBDC and the choice of CBDC interest rate policy do not significantly influence how the economy reacts to shocks. Assenmacher et al. (2023) incorporate NK frictions in an NM model and, similar to our paper, study how a CBDC could change the response of the economy to macroeconomic shocks. They find that having a CBDC does not markedly change the model's response to macroeconomic shocks, and it usually moderates and smooths their transmission to key metrics like output and inflation. In contrast to the last two aforementioned papers, we provide nuances for the response of the economy in the presence of a CBDC and show that details matter substantively, i.e., the type of shocks, the monetary policy framework, and the balance sheet quantity rules, for macroeconomic outcomes.

In an extension of Minesso et al. (2022), Assenmacher et al. (2024) include financial frictions and occasionally binding constraints in a two-country DSGE model. They find that the transition from a non-CBDC regime to one with a CBDC triggers an initial surge in demand for CBDC and money, leading to the displacement of bank deposits and a subsequent decline in investment, consumption, and output. These results are consistent with empirical evidence provided by Bidder et al. (2024) from European era of slow disintermediation of the banking system. These papers focus mostly on the transitional dynamics of introducing a CBDC and not so much on the transmission channels of standard monetary policy shocks or balance sheet quantity shocks in the presence of a CBDC.[4]

[4]We organize related literature into four categories, noting that we cannot do justice to all. The first category, CBDC and banking, examines the impact of CBDC on the traditional banking system: Andolfatto (2021), Keister and Sanches (2023), and Chiu et al. (2023), as well as Garratt et al. (2022), Fernández-Villaverde et al. (2021), Benigno et al. (2022), Niepelt (2020), and Williamson (2022a,b). The second category, CBDC and monetary policy, examines the macroeconomic consequences of CBDCs in DSGE, NM, and other models: Barrdear and Kumhof (2022), Assenmacher et al. (2023), Davoodalhosseini (2022), and Benigno and Benigno (2021). The third category, CBDC and financial stability, investigates the implications of a CBDC on the riskiness of banks, financial intermediation, the coordination runs faced by banks, and

Finally, our paper is related to another strand of the literature, standard macroeconomic models that take financial frictions and the role of liquidity seriously. Some related models include Gertler and Karadi (2011), Gertler and Karadi (2015), Gertler and Kiyotaki (2010), Sims and Wu (2021), Krishnamurthy and Vissing-Jorgensen (2012), Benigno and Benigno (2021), and Bhattarai and Neely (2022). See also Ahnert et al. (2022), Chapman et al. (2023), and Davoodalhosseini and Rivadeneyra (2020) for general discussions and literature reviews regarding electronic monies and CBDCs.

2 Model

We use a relatively simple model to clarify the main channels through which CBDC affects the economy. The model consists of households, firms, banks and the central bank (which includes government). We describe them individually below. The time is discrete, $t = 0, 1, ...$, and goes forever. Banks issue deposits to households and make loans to firms. Households use these deposits as payment means, in addition to the CBDC issued by the government. In our model, CBDC can be understood as central bank accounts accessible to households that earn interest (if the central bank so decides) and can be used for payments.[5] In this version of the model, we do not include cash to simplify the analysis and clarify the mechanisms in action.[6] Finally, the central bank issues reserves solely available to banks to back deposits.

Banks in our model are relatively simple and somewhat similar to the banks in Piazzesi et al. (2019). However, a key difference is that we incorporate the credit channel of monetary policy. In our model, firms depend on banks for their capital funding, meaning that changes in the cost of loans (stemming from shifts in the cost of reserves or deposits for banks) impact the supply side of the economy. Thus, a feedback loop exists from bank credit to firms' investment in capital and overall aggregate supply. In contrast, the banks in Piazzesi et al. (2019) invest in assets that do not affect the supply side of the economy.

The production side in our model follows standard NK models with financial frictions, e.g, Gertler and Karadi (2011) and Gertler and Karadi (2015).

2.1 Households

The households in our model consume a final good for which they need to use means of payments, including deposits and CBDC. We use the money-in-the-utility-function frame-

payment systems, and whether a perfectly safe bank could be a substitute for CBDC: Fernández-Villaverde et al. (2021), Keister and Sanches (2023), Williamson (2022a), and Chiu et al. (2020). The fourth and final category, digital currency design and platforms, explores the design features of digital currencies and CBDCs, and some papers consider these currencies as competing platforms: Chiu and Wong (2015), Chiu and Koeppl (2019), Kahn et al. (2022), Brunnermeier et al. (2022), and Cheng et al. (2024).

[5] Although we adopt this interpretation for the sake of simplicity, we are silent on the technology through which CBDC is introduced. The verification of the identity of the CBDC holder may or may not be required. CBDC can even be token based. We do not discuss details here. See the literature review by Chapman et al. (2023) for further discussion on the design of a CBDC.

[6] In Appendix C, we add cash to the model and rewrite the equations affected by this addition. That will help us to study other CBDC designs like a cash-like or universal CBDC. In Appendix D, we expand on some special cases of the model.

work and obtain a demand function for means of payments.[7] Using this model , one can study the implications of different design features of a CBDC for macroeconomic outcomes. Preferences at time t over consumption good, means of payments, and labor are given by

$$U\left(C_t, \frac{D_t}{P_t}, \frac{F_t}{P_t}, H_t\right) = \frac{1}{1-\frac{1}{\sigma}}\left(C_t^{1-\frac{1}{\eta}} + \omega_D \mathbf{Liq}_t^{1-\frac{1}{\eta}}\right)^{\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\eta}}} - \frac{\psi}{1+\varphi}H_t^{1+\varphi},$$

where $\mathbf{Liq}_t$ is a liquidity aggregator,

$$\mathbf{Liq}_t \equiv \left((D_t/P_t)^{1-\frac{1}{\upsilon}} + \frac{\omega_{FD}}{\omega_D}(F_t/P_t)^{1-\frac{1}{\upsilon}}\right)^{\frac{1}{1-\frac{1}{\upsilon}}},$$

and D_t and F_t denote the nominal deposit and CBDC balances and bear interest rates i_t^D and i_t^F, respectively; H_t denotes the units of labor supplied and P_t is the price level. As in Piazzesi et al. (2019), the utility function features η, which denotes the intratemporal elasticity of substitution between consumption and the liquidity aggregator, and σ denotes the intertemporal elasticity of substitution between the consumption bundle today and tomorrow. Several points are in order regarding this utility function. As long as $\sigma = \eta$, the utility is separable in consumption and the liquidity aggregator. If $\sigma > \eta$ ($\sigma < \eta$), consumption and the liquidity aggregator are complements (substitutes).

There are two design features of a CBDC. These design features are determined by the CBDC technology, and the government might be able to affect them. First, υ denotes the elasticity of substitution between deposits and the CBDC. As υ increases, deposits and the CBDC become better substitutes and remain substitutes as long as $\upsilon > \eta$. Second, the term $\frac{\omega_{FD}}{\omega_D}$ captures the relative transaction costs for households. For example, when the CBDC and deposits are perfect substitutes ($\upsilon = \infty$), $\frac{\omega_{FD}}{\omega_D} = 1.01$ implies that CBDC is 1% more effective than deposits in payments.

This specification for CBDC is new in the literature and allows for a rich set of design features. As a result of this specification, the model nests several special designs for CBDCs that have been discussed in the literature. For example, a deposit-like CBDC (as in Chiu and Davoodalhosseini (2023) and Keister and Sanches (2023)) is a perfect substitute with bank deposits: $\upsilon = \infty$. One can map the market shares and merchants' acceptability to different values of υ, ω_D, and ω_{FD} and then experiment with different designs of the CBDC.

A household's maximization problem at date 0 is given by

$$\max \quad \boldsymbol{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, \frac{D_t}{P_t}, \frac{F_t}{P_t}, H_t\right)$$

$$\begin{aligned} \text{s.t.} \qquad & P_t C_t + D_t + F_t + S_t \\ = \quad & W_t H_t - T_t + \Pi_t + D_{t-1}\left(1 + i_{t-1}^D\right) + F_{t-1}\left(1 + i_{t-1}^F\right) + S_{t-1}\left(1 + i_{t-1}^S\right). \end{aligned}$$

[7] As summarized in Schmitt-Grohé and Uribe (2010), there are various ways in the literature to motivate the money demand function: money in the utility function (Sidrauski (1967)), cash-in-advance constraint (Lucas Jr (1982)), shopping time technology (Kimbrough (1986)), or a transactions-cost technology (Feenstra (1986)).

The budget constraint is standard. Households receive wage income, W_tH_t, and profits of the firms, Π_t, and pay taxes, T_t, to the government (which can be negative). Households can invest in riskless private or public bonds, denoted by S_t with nominal interest rate i_t^S. They receive interest on deposits, CBDC and bond holdings.

2.1.1 Households' Optimality Conditions

Denote by λ_t the Lagrangian multiplier associated with the budget constraint. Households' optimality conditions can be written as follows:

$$
\begin{aligned}
C:&\quad \frac{U_{C,t}}{P_t} = \lambda_t, \\
J:&\quad \frac{U_{J,t}}{P_t} = \lambda_t - \beta \boldsymbol{E}_t \lambda_{t+1}\left(1+i_t^J\right) \text{ for } J \in \{D, F\}, \\
S:&\quad \lambda_t = \beta \boldsymbol{E}_t \lambda_{t+1}\left(1+i_t^S\right), \\
H:&\quad U_{H,t} = -\lambda_t W_t.
\end{aligned}
$$

The optimality conditions for S and each $J \in \{D, F\}$ yields

$$
\frac{U_{J,t}}{P_t} = \lambda_t - \beta \boldsymbol{E}_t \lambda_{t+1}\left(1+i_t^S - (i_t^S - i_t^J)\right) = \beta \boldsymbol{E}_t \lambda_{t+1}\left(i_t^S - i_t^J\right) = \lambda_t \frac{i_t^S - i_t^J}{1+i_t^S}.
$$

The money demand function for deposits and the CBDC are given by

$$
\frac{i_t^S - i_t^D}{1+i_t^S} \geq \omega_D V_{D,t}^{\frac{1}{\upsilon}} V_{FD,t}^{-\frac{1}{\upsilon}+\frac{1}{\eta}} \text{with equality if } D_t > 0, \tag{1}
$$

$$
\frac{i_t^S - i_t^F}{1+i_t^S} \geq \omega_{FD} V_{F,t}^{\frac{1}{\upsilon}} V_{FD,t}^{-\frac{1}{\upsilon}+\frac{1}{\eta}} \text{with equality if } F_t > 0, \tag{2}
$$

where

$$
\text{Velocity}: \quad V_{J,t} \equiv \frac{P_t C_t}{J_t} \text{ for } J \in \{D, F\},
$$

and

$$
V_{FD,t} \equiv \left(V_{D,t}^{\frac{1}{\upsilon}-1} + \frac{\omega_{FD}}{\omega_D} V_{F,t}^{\frac{1}{\upsilon}-1}\right)^{\frac{1}{\frac{1}{\upsilon}-1}}.
$$

The money demand function in our paper is a generalization of that in Piazzesi et al. (2019). In their paper, there is neither a competition between the means of payments nor a rich set of substitution/complementarity patterns between them. To understand these demand functions, we focus on (2). Understanding the other one is straightforward.

The LHS of (2) shows the marginal cost (i.e., the opportunity cost) of holding a unit of CBDC. By holding CBDC, households forgo the interest that they could get by holding riskless bonds, i^S. Instead, they receive interest on the CBDC, i^F. The RHS shows the marginal benefit of holding a unit of CBDC, which comprises the benefit of using CBDC in conjunction with deposits. The marginal benefit is equal to a combination of two terms, including the velocity of the CBDC augmented by various elasticities and weights. By velocity we mean the standard definition, i.e., how much consumption can be purchased using one unit of money in circulation.

Now consider the first term on the RHS of (2). The marginal benefit of CBDC in the deposit bundle increases in the velocity of the CBDC. A higher υ means that deposits are a better substitute for CBDC, so the marginal benefit of a unit of CBDC is lower. The marginal benefit of CBDC also depends on the term V_{FD} (except for the special case of $\upsilon = \eta$). If CBDC and deposits are substitutes, the marginal benefit decreases in V_{FD}, which is the inverse of a CES aggregator between velocities of deposits and CBDC. Finally, note that this equation is independent of σ, implying that the demand functions derived here are unchanged whether the utility function is separable or not.

Using (1) and (2), we can relate the opportunity cost of holding CBDC with that of deposits:

$$i_t^S - i_t^F = \frac{\omega_{FD}}{\omega_D} \left(\frac{D_t}{F_t} \right)^{\frac{1}{\upsilon}} \left(i_t^S - i_t^D \right). \tag{3}$$

Intuitively, one can think that once CBDC is introduced, it can serve the transactions in addition to deposits. The coefficient, $\frac{\omega_{FD}}{\omega_D} \left(\frac{D_t}{F_t} \right)^{\frac{1}{\upsilon}}$, on RHS captures the relative usefulness of CBDC relative to deposits. This is multiplied by the marginal cost of using deposits. Therefore, the RHS of (3) captures the overall benefits of the CBDC in deposit transactions. Finally, note that i^D and i^F can go below zero depending on the liquidity service they provide to households.

The optimality condition for the labor supply is given by

$$C_t^{\frac{1}{\sigma}} \psi H_t^{\varphi} = Q_t^{\frac{\eta}{\sigma}-1} \frac{W_t}{P_t}, \tag{4}$$

where

$$Q_t \equiv \left(1 + \omega_D V_{FD,t}^{\frac{1-\eta}{\eta}} \right)^{\frac{1}{1-\eta}}. \tag{5}$$

Equation (4) describes a household's labor supply decision and is similar to that obtained by Piazzesi et al. (2019) except that the definition of Q has been modified. This equation states that the marginal rate of substitution between consumption and labor supply is simply equal to the relative price of the two, which is equal to the real wage. For the case of separable utility ($\eta = \sigma$), the equation is self-explanatory. In the more general case of non-separable utility ($\eta \neq \sigma$), the marginal rate of substitution includes another term, Q_t. To provide a more concrete explanation, consider the case where consumption and means of payments are complements ($\sigma > \eta$), the case that resembles closely the models with cash-in-advance constraints. Empirical studies also confirm that $\sigma > \eta$ is indeed the case. As the nominal interest rate increases, the opportunity cost of holding money rises, which works effectively as a tax on consumption and consequently a subsidy on leisure. Therefore, agents reduce their labor supply as the opportunity cost of holding money increases. (See Chapter 3 of Walsh (2017)). To see this effect clearly in our model, simply put $\upsilon = \eta$ as a special case, which captures the case that all means of payments have the same elasticity with respect to change in interest rates. As the nominal interest rate i^S rises, the velocities rise too (from the demand functions for means of payments), implying that Q increases. Therefore, the supply of labor, H, should fall from (4), all else being equal.[8]

The Euler equation for illiquid bonds can be written as

$$\beta \boldsymbol{E}_t \left[\left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{\eta}{\sigma}-1} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\sigma}} \frac{P_t}{P_{t+1}} \right] (1 + i_t^S) = 1. \tag{6}$$

The optimal choice of an illiquid bond requires the rate of return on the illiquid bond discounted by the stochastic discount factor be equal to one. Again, the equation is standard in the separable case. In the non-separable case, particularly when $\sigma > \eta$, as the nominal interest increases, the opportunity cost of holding money increases. Therefore, the agent discounts the future less (as mentioned above in the discussion about Q), leading to less consumption today.

To put everything together regarding a household's optimality conditions: (6) is a generalization of a standard Euler equation in a case where consumption and means of payments are complements. This equation determines the relationship between the nominal interest rate on illiquid bonds with the consumption of housholds. Equations (1) and (2) then relate the velocity of different means of payments with the wedge between the nominal rate on illiquid bonds and the nominal interest rate on that means of payment. This wedge arises because of the liquidity that these means of payments provide to agents.

2.1.2 The Role of CBDC

CBDC serves two roles in this model. First, it changes the demand for deposits, and the change depends on different parameters including the elasticity of substitution between deposits and the CBDC. Second, it provides liquidity to households just like deposits, so it changes the incentives to supply labor and to consume through labor supply and Euler equations. Now, we elaborate on them.

First, write V_D and V_F as functions of V_{FD},[9] then use (3) to solve for V_{FD}:

$$V_{FD} = \omega_D^{\frac{\eta}{\upsilon-1}} \left(\omega_D^{\upsilon} \left(\frac{i^S - i^D}{1+i^S} \right)^{1-\upsilon} + \omega_{FD}^{\upsilon} \left(\frac{i^S - i^F}{1+i^S} \right)^{1-\upsilon} \right)^{-\frac{\eta}{\upsilon-1}}. \tag{7}$$

[8]This channel is referred to as the "cost channel" in some DSGE papers, although it is often given relatively little attention. In contrast, models that incorporate cash-in-advance constraints or follow the NM approach consider this channel central to their frameworks. For example, in NM models, an increase in the opportunity cost of means of payments leads buyers to bring less real balances to the decentralized market. This means that sellers work and produce less, so the output falls.

[9]We have $V_D^{\frac{1-\upsilon}{\upsilon}} = \omega_D^{\upsilon-1} \left(\frac{i^S - i^D}{1+i^S} \right)^{1-\upsilon} V_{FD}^{-\frac{\upsilon-\eta}{\eta}\frac{1-\upsilon}{\upsilon}}$ and $V_F^{\frac{1-\upsilon}{\upsilon}} = \omega_{FD}^{\upsilon-1} \left(\frac{i^S - i^F}{1+i^S} \right)^{1-\upsilon} V_{FD}^{-\frac{\upsilon-\eta}{\eta}\frac{1-\upsilon}{\upsilon}}$.

Using the equation for V_D again, we obtain

$$V_D^{\frac{1-\upsilon}{\upsilon}} = \omega_D^{\upsilon-\frac{\eta}{\upsilon}} \frac{\left(\frac{i^S-i^D}{1+i^S}\right)^{1-\upsilon}}{\left(\omega_D^{\upsilon}\left(\frac{i^S-i^D}{1+i^S}\right)^{1-\upsilon} + \omega_{FD}^{\upsilon}\left(\frac{i^S-i^F}{1+i^S}\right)^{1-\upsilon}\right)^{\frac{\upsilon-\eta}{\upsilon}}}. \tag{8}$$

This equation is a modified demand function for deposits in the presence of a CBDC. In the case that a CBDC does not exist ($\omega_{FD} = 0$), this equation reduces to $V_D = \omega_D^{-\eta}\left(\left(i^S - i^D\right)/\left(1 + i^S\right)\right)^{\eta}$. Like a typical money demand equation, (8) states that the velocity of deposits depends on the opportunity cost of deposits (the numerator), but this opportunity cost should be adjusted by an aggregator, which depends on the opportunity costs of both deposits and the CDBC (the denominator). For example, when the CBDC and deposits are perfect substitutes ($\upsilon \to \infty$), then the velocity of deposits would be too large (i.e., no demand for deposits) if $i^F > i^D$ and would be unaffected by CBDC if $i^F < i^D$.

Second, we calculate Q using (7):

$$Q = \left(1 + \omega_D^{\frac{\upsilon-\eta}{\upsilon-1}}\left(\omega_D^{\upsilon}\left(\frac{i^S - i^D}{1 + i^S}\right)^{1-\upsilon} + \omega_{FD}^{\upsilon}\left(\frac{i^S - i^F}{1 + i^S}\right)^{1-\upsilon}\right)^{-\frac{1-\eta}{\upsilon-1}}\right)^{\frac{1}{1-\eta}}. \tag{9}$$

This equation captures the fact that the CBDC provides real balances like deposits. A higher opportunity cost of the CBDC increases Q, regardless of the elasticity of substitution between deposits and the CBDC (υ) or the intratemporal substitution between consumption and money composite (η). A higher Q implies that the adjusted real wage in (4) falls, leading to lower consumption. Note, however, that the magnitude of the effect depends on both υ and η.

2.2 Bank Problem

Banks are short-lived; they are born at the beginning of period t and die at the end of period $t + 1$. They are perfectly competitive. It's easy to allow for market power, but we abstract from it to focus on the main contribution of the paper regarding monetary policy transmission. On the liability side, they can issue liquid deposits, D_t, which are used in transactions by households. They can also issue illiquid bonds, A_t, which are perfectly safe and cost less to adjust.[10] A_t can be negative too. On the asset side, they can buy reserves from the central bank, M_t, or they can invest in real claims on capital, b_t. The interest rate on reserves is denoted by i_t^M, and the interest on deposits is equal to i_t^D. The cost of issuing illiquid bonds is equal to i^S, which is the nominal interest rate on government bonds too. Public and private bonds are similar as there is no default for any of them here.

Banks are subject to

$$\begin{aligned} N_t &= M_t + P_t b_t - D_t - A_t, \\ D_t &\leq \ell(M_t + \rho P_t b_t). \end{aligned}$$

[10] In Appendix B, we discuss the role of different model ingredients, for example, what if banks cannot hold illiquid bonds.

The first constraint is the bank's balance sheet identity. The second constraint, similar to that in Piazzesi et al. (2019), is a leverage constraint. We impose that $\ell < 1$ and $\rho < 1$. The latter means that other assets have a lower quality compared with reserves to be used as collateral.

Bank profit at time $t+1$ is denoted by Ψ_{t+1} and given by

$$\Psi_{t+1} = P_t b_t \left(1 + i^K_{t+1}\right) + M_t \left(1 + i^M_t\right) - \left(1 + i^D_t\right) D_t - \left(1 + i^S_t\right) A_t.$$

Banks maximize the expected value of discounted profits minus equity, which is given by the following for time t:

$$\begin{aligned} \mathcal{R}_t &= \boldsymbol{E}_t \left\{\bar{\Lambda}_{t,t+1}\Psi_{t+1}\right\} - N_t \\ &= \boldsymbol{E}_t \left\{\bar{\Lambda}_{t,t+1} \left[\begin{array}{c} P_t b_t \left(1 + i^K_{t+1}\right) + M_t \left(1 + i^M_t\right) \\ - \left(1 + i^D_t\right) D_t - \left(1 + i^S_t\right) A_t \end{array} \right]\right\} - N_t, \end{aligned}$$

where $\bar{\Lambda}_{t,t+1} \equiv \beta U_{C,t+1}/(\pi_{t+1} U_{C,t})$ is the nominal stochastic discount factor.

We first eliminate N from the problem and then write the Lagrangian for banks' problem as

$$\begin{aligned} &\boldsymbol{E}_t \left\{\bar{\Lambda}_{t,t+1} \left[\begin{array}{c} P_t b_t \left(1 + i^K_{t+1} + M_t \left(1 + i^M_t\right)\right) \\ - \left(1 + i^D_t\right) D_t - \left(1 + i^S_t\right) A_t \end{array} \right]\right\} + A_t - M_t - P_t b_t + D_t \\ &+\lambda_t \left(-D_t + \ell M_t + \ell \rho P_t b_t\right), \end{aligned}$$

where λ_t is the Lagrangian multiplier associated with the leverage constraint.

Write optimality conditions and manipulate them to obtain

$$\boldsymbol{E}_t \left\{\bar{\Lambda}_{t,t+1} \left(i^S_t - i^D_t\right)\right\} = \frac{E_t \left\{\bar{\Lambda}_{t,t+1} \left(i^S_t - i^M_t\right)\right\}}{\ell} = \frac{E_t \left\{\bar{\Lambda}_{t,t+1} \left(i^S_t - i^K_{t+1}\right)\right\}}{\ell \rho} = \lambda_t,$$

$$\boldsymbol{E}_t \left\{\bar{\Lambda}_{t,t+1} \left(1 + i^S_t\right)\right\} = 1.$$

Note that the terms in the expectation are all determinate at time t except the interest on the claims on capital and the discount factor Λ_{t+1}, so the conditions can be more simply written as

$$\frac{i^S_t - i^D_t}{1 + i^S_t} = \frac{i^S_t - i^M_t}{\ell \left(1 + i^S_t\right)} = \frac{i^S_t - \frac{\boldsymbol{E}_t\left(\bar{\Lambda}_{t+1} i^K_{t+1}\right)}{\boldsymbol{E}_t \bar{\Lambda}_{t+1}}}{\ell \rho \left(1 + i^S_t\right)} = \lambda_t.$$

Note that in equilibrium, we must have $i^S_t - i^D_t > 0$ from the money demand equation, so the constraint is always binding, i.e., $\lambda_t > 0$.

These equations relate to the spread of deposits, reserves and the return on capital. A higher ℓ implies that the assets are better in backing liabilities, so the interest on reserves and the rate of return on capital both decrease.

Sometimes it is useful to write this equation in terms of only the real return on capital, so we will sometimes use $1 + i^K_{t+1} = \left(1 + r^K_{t+1}\right) \pi_{t+1}$, where r^K_{t+1} is the real interest rate on capital and π_{t+1} is the inflation rate at $t+1$.

2.3 Production

The production side of the economy is standard and borrowed from standard New Keynsian models with financial constraints (a la Gertler and Kiyotaki (2010), Gertler and Karadi (2015) and Gertler and Karadi (2011)). There are three types of non-financial firms: (i) intermediate goods producers, (ii) capital producers, and (iii) monopolistically competitive retailers subject to nominal price rigidity.

2.3.1 Intermediate Goods Producers

Intermediate goods producers use capital and labor according to the following production function:

$$Y_t = \digamma_t K_t^{\alpha} L_t^{1-\alpha},$$

where $\digamma_t$, K_t, and L_t denote productivity, capital and labor.

These producers buy new capital from capital-producing firms at the begining of each period and then sell the depreciated capital at the end of the period. The producers do not have funds, so they issue equity (or a perfectly state-contingent debt) to banks and pay back at the end of the period. The price of the claims are denoted by X_t, so

$$X_t K_{t+1} = X_t b_t.$$

The evolution of capital stock can be written as follows:

$$K_{t+1} = I_t + (1-\delta)\xi_t K_t,$$

where ξ_t denotes the capital quality shock and δ is the depreciation rate.

There are no frictions on the side of the intermediate goods producers when financed by banks. As defined earlier, the rate of return on banks' capital investment is $1 + i_{t+1}^K$, so the producer's maximization problem is given by

$$\max_{L_t} \left\{ p_{mt} \digamma_t K_t^{\alpha} L_t^{1-\alpha} - w_t L_t + \xi_t X_t (1-\delta) K_t \right\},$$

where $p_{mt} = \frac{P_{mt}}{P_t}$ is the real price of intermediate good. Similarly, $w_t \equiv \frac{W_t}{P_t}$ is the real wage. In a competitive market for labor, the demand for labor is given by

$$w_t = p_{mt}(1-\alpha)\frac{Y_t}{L_t}.$$

The profit per unit of capital is given by $Z_t = \alpha p_{mt} \frac{Y_t}{K_t}$.

The maximized value is the revenue of the firm after paying wages and selling the used capital at the end of the period. Given the initial investment of $X_{t-1}K_t$, the return on investment is given by

$$1 + r_t^K = \frac{\max_{L_t} \left\{ p_{mt} \digamma_t K_t^{\alpha} L_t^{1-\alpha} - w_t L_t + X_t \xi_t (1-\delta) K_t \right\}}{X_{t-1} K_t}.$$

The rate of return on capital is thus summarized as follows:

$$1 + r_t^K = \frac{Z_t + (1-\delta)\xi_t X_t}{X_{t-1}}. \tag{10}$$

This denotes the payoff of the firm at time t per unit of capital divided by the price of the claims issued to the banks.

2.3.2 Capital Producers

The capital producer turns the final output into capital, denoted by I, subject to adjustment costs, denoted by f function:

$$\max \boldsymbol{E}_t \sum_{\tau=t}^{\infty} \Lambda_{t,\tau} \left\{ X_\tau^i I_\tau - \left[1 + f\left(\frac{I_\tau}{I_{\tau-1}} \right) \right] I_\tau \right\},$$

where $\Lambda_{t,t+i} \equiv \beta^i \frac{U_{c,t+i}}{U_{c,t}}$ denotes the stochastic discount factor. The price of capital goods in terms of the final good is given by the following:

$$X_t = 1 + f\left(\frac{I_t}{I_{t-1}} \right) + \frac{I_t}{I_{t-1}} f'\left(\frac{I_t}{I_{t-1}} \right) - \boldsymbol{E}_t \Lambda_{t,t+1} \left(\frac{I_{t+1}}{I_t} \right)^2 f'\left(\frac{I_{t+1}}{I_t} \right),$$

which is equal to the marginal cost of investment goods production.

2.3.3 Retailers

Consumption goods is the CES composite of intermediate goods with elasticity ϵ. That is,

$$C_t \equiv \left(\int_0^1 C_t(i)^{1-\frac{1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}.$$

Firm $i \in [0,1]$ produces output $y_t(i) = h_t(i)$ from the intermediate good using linear technology. Firm i sets prices $P_t(i)$ for its good facing demand $y_t(i) = y_t \left(P_t(i)/P_t \right)^{-\epsilon}$. The profit at period t is given by $P_\tau(i) h_\tau(i)/P_\tau - d\left(P_\tau(i)/P_{\tau-1}(i) \right) - P_{m\tau} h_\tau(i)$. The firm chooses $P_t(i)$ and $h_t(i)$ to maximize expected discounted profits. We substitute out $y_t(i)$ to write the maximization problem as

$$\max \boldsymbol{E}_t \sum_{\tau=t}^{\infty} \Lambda_{t,\tau} \left(\frac{P_\tau(i)}{P_\tau} y_t \left(\frac{P_\tau(i)}{P_\tau} \right)^{-\epsilon} - d\left(\frac{P_\tau(i)}{P_{\tau-1}(i)} \right) - p_{m\tau} y_\tau \left(\frac{P_\tau(i)}{P_\tau} \right)^{-\epsilon} \right).$$

The optimality conditions require

$$\left[(1-\epsilon) \frac{P_t(i)}{P_t} + \epsilon p_{m\tau} \right] \left(\frac{P_t(i)}{P_t} \right)^{-\epsilon} y_t - \frac{P_t(i)}{P_{t-1}(i)} d'\left(\frac{P_t(i)}{P_{t-1}(i)} \right) + \boldsymbol{E}_t \left[\frac{P_{t+1}(i)}{P_t(i)} \Lambda_{t,t+1} d'\left(\frac{P_{t+1}(i)}{P_t(i)} \right) \right] = 0.$$

Assuming $d(x) = \frac{\kappa}{2}(x-1)^2$, we have $d'(x) = \kappa(x-1)$. Imposing symmetry across firms' responses (i.e., $P_t = P_t(i)$), we obtain

$$\left[\frac{\epsilon-1}{\epsilon} - p_{m\tau}\right]\frac{\epsilon Y_t}{\kappa} + \frac{P_t}{P_{t-1}}\left(\frac{P_t}{P_{t-1}} - 1\right) = \boldsymbol{E}_t\left[\Lambda_{t,t+1}\frac{P_{t+1}}{P_t}\left(\frac{P_{t+1}}{P_t} - 1\right)\right]. \tag{11}$$

This is a standard Philips curve.

We assume zero capital adjustment costs for the benchmark model (not for calibration), so we must have

$$X_t = 1.$$

For the case with no capital quality shock, that is, $\xi_t = 1$ for all t, (10) gives

$$1 + r_t^K = \alpha p_{mt}\frac{Y_t}{K_t} + 1 - \delta, \tag{12}$$

or equivalently, $p_{mt} = \frac{r_t^K + \delta}{\alpha}\ \frac{K_t}{Y_t}$.

2.3.4 Market Clearing Conditions

The market clearing condition for the final good is given by

$$Y_t = C_t + \left(1 + f\left(\frac{I_t}{I_{t-1}}\right)\right) I_t + \frac{\kappa}{2}\left(\frac{P_t}{P_{t-1}} - 1\right)^2. \tag{13}$$

Labor supply and demand equations together imply

$$Q_t^{1-\frac{\eta}{\sigma}} C_t^{\frac{1}{\sigma}} \psi H_t^{\varphi} = w_t = p_{mt}(1-\alpha)\frac{Y_t}{L_t}.$$

We now can obtain p_{mt}:

$$p_{mt} = Q_t^{1-\frac{\eta}{\sigma}} C_t^{\frac{1}{\sigma}} \psi \frac{L_t^{1+\varphi}}{(1-\alpha)Y_t},$$

where we used the labor market clearing condition, $H_t = L_t$. This equation gives us H_t.

2.3.5 Government and Central Bank

Here, we describe the government's and the central bank's budget constraints as well as the equilibrium condition for the illiquid bond issued by the government and banks.

The government's budget constraint is given by

$$T_t + S_t^G + TR_t = (1 + i_{t-1}^S)S_{t-1}^G,$$

where TR_t denotes the nominal transfer from the central bank to the treasury, T_t denotes the net lump sum tax levied on households, and S_t^G denotes the supply of safe illiquid bonds by the government. The central bank's budget constraint is given by

$$F_t + M_t = TR_t + (1 + i_{t-1}^F)F_{t-1} + (1 + i_{t-1}^M)M_{t-1}.$$

The illiquid bonds are demanded by households and supplied by banks and the government, so the market clearing condition for these bonds is given by

$$S_t^G + A_t = S_t.$$

We don't impose any constraint on, T_t, TR_t, and S_t^G, so the equations in this subsection can be solved in a separate block from the rest of equations of the model. That is, T_t, TR_t, and S_t^G are adjusted by these three equations after the determination of equilibrium variables including S_t and A_t. We abstract from matters related to fiscal policy and the relationship between monetary and fiscal authority.

3 Steady-State Analysis

In this section, we characterize the steady-state equilibrium with a zero inflation rate. In the next section, we analyse the effects of various shocks to this economy starting from the zero-inflation rate steady state.

We assume that the central bank sets the interest rates on the CBDC and reserves, i^M, i^F. An alternative policy could be that the central bank sets the real quantity of reserves, for example. We set the aggregate price level to 1. We drop the subscript t to show the steady-state levels. The unknown variables of the model are

- Output, consumption and labor: Y, C, L
- Deposits, CBDC and reserves balances: D, F, M
- Real assets: b
- Rates: i^K, i^D

Note that the nominal and real interest rates are equal because the inflation rate is zero, i.e., $i^K = r^K$ and $i^D = r^D$. We now derive the steady-state values:

$$\begin{aligned} \text{Intermediate good price:} \quad & p_m = \frac{\epsilon - 1}{\epsilon}\ , \\ \text{Illiquid bond demand:} \quad & \beta\left(1 + i^S\right) = 1. \end{aligned}$$

Block 1: Given i^M, we can solve for i^K and i^D:

$$\frac{i^S - i^D}{1 + i^S} = \frac{i^S - i^M}{\left(1 + i^S\right)\ell} = \frac{i^S - i^K}{\left(1 + i^S\right)\rho\ell}. \tag{14}$$

Block 2: Given i^D from block 1 and i^F from policy, we can pin down V_D and V_F:

$$\text{Deposit demand:} \quad \frac{i^S - i^D}{1 + i^S} = \omega_D V_D^{\frac{1}{\upsilon}} V_{FD}^{-\frac{1}{\upsilon}+\frac{1}{\eta}}, \tag{15}$$

$$\text{CBDC demand:} \quad \frac{i^S - i^F}{1 + i^S} = \omega_{FD} V_F^{\frac{1}{\upsilon}} V_{FD}^{-\frac{1}{\upsilon}+\frac{1}{\eta}}, \tag{16}$$

$$\text{Velocity of D:} \quad V_{FD} = (V_D^{\frac{1}{\upsilon}-1} + \frac{\omega_{FD}}{\omega_D} V_F^{\frac{1}{\upsilon}-1})^{\frac{1}{\frac{1}{\upsilon}-1}}, \tag{17}$$

Moreover, we have

$$b = K = \frac{I}{\delta},$$

and

$$Q \equiv \left(1 + \omega_D V_{FD}^{\frac{1}{\eta}-1}\right)^{\frac{1}{1-\eta}}.$$

Block 3: Given i^K (from block 1) and Q (from block 2), the following four equations pin down Y, L, C and b :

$$\begin{aligned}
Y &= C + \delta K, \\
p_m &= \frac{\epsilon - 1}{\epsilon} = \frac{\left(i^K + \delta\right) K}{\alpha Y} \to K = \frac{\epsilon - 1}{\epsilon} \frac{\alpha Y}{i^K + \delta}, \\
Y &= A K^{\alpha} L^{1-\alpha}, \\
\frac{\epsilon - 1}{\epsilon} Y &= Q^{1-\frac{\eta}{\sigma}} C^{\frac{1}{\sigma}} \psi \frac{L^{1+\varphi}}{1-\alpha}.
\end{aligned} \tag{18}$$

We can now calculate Y as a function of C and K and then use the market clearing condition:

$$Y^{\frac{1+\varphi}{1-\alpha}-1} = \frac{\epsilon - 1}{\epsilon} \frac{(1-\alpha) A^{\frac{1+\varphi}{1-\alpha}} K^{\alpha \frac{1+\varphi}{1-\alpha}}}{Q^{1-\frac{\eta}{\sigma}} C^{\frac{1}{\sigma}} \psi},$$

$$Y = C + \delta K \to C = Y \left(1 - \frac{\epsilon - 1}{\epsilon} \frac{\alpha \delta}{i^K + \delta}\right). \tag{19}$$

Given that K and C are now given in terms of Y from (18) and (19), we obtain

$$Y^{\varphi + \frac{1}{\sigma}} = \frac{\alpha^{\frac{\alpha(1+\varphi)}{1-\alpha}} (1-\alpha) A^{\frac{1+\varphi}{1-\alpha}} \left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}-1}}{\psi Q^{1-\frac{\eta}{\sigma}} \left(1 - \frac{\epsilon-1}{\epsilon} \frac{\alpha\delta}{i^K+\delta}\right)^{\frac{1}{\sigma}} \left(i^K + \delta\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}}}. \tag{20}$$

This equation is important because it gives Y simply as a function of Q and i^K. Given Y, one can easily solve for L, C and b.

First, consider the case of the **separable utility** function, $\eta = \sigma$. The output is only a function of i^K, which is pinned down by i^M, given that the bank's constraint is binding. In this case, the interest rate on the CBDC changes only the velocity of money but does not affect output, consumption or investment. The intuition is simple. With a separable utility function, there is no complementarity between money balances and consumption, so the typical channel that exists in CIA models would be absent. In this case, an additional unit of money does not help in terms of actual consumption, so the only channel for transmission of monetary policy is through the change in the opportunity cost of lending for banks. As the central bank increases i^M, the opportunity cost of lending rises, so banks tend to hold more reserves and lend less. Lemma 1 states conditions under which investment, capital, output and consumption decrease.

Lemma 1 *Assume $\eta = \sigma$. Then,*

(i) Real variables depend on policy only through i^K. So, if i^M is kept fixed, a change in elasticities, ω's, or the CBDC rate does not change real variables.

(ii) An increase in i^M decreases output, capital and investment; also consumption decreases if δ is sufficiently close to zero.

Proof. (i): obvious from (20).
(ii): note that an increase in i^M increases i^K through (14), which **decreases output** through (20) and capital through (18), and consequently I decreases because I=δK. For C, combine (19) and (20) to obtain

$$\begin{aligned} C &= const. \left(1 - \frac{\epsilon - 1}{\epsilon} \frac{\alpha\delta}{i^K + \delta}\right)^{\frac{\sigma\varphi}{1+\sigma\varphi}} \left(i^K + \delta\right)^{\frac{-\alpha(1+\varphi)\sigma}{(1-\alpha)(1+\sigma\varphi)}} \\ &\rightarrow \frac{\partial \ln C}{\partial i^K} \left(\frac{i^K + \delta}{\sigma}\right) \left(\frac{1 + \sigma\varphi}{\varphi}\right) = \frac{1}{\frac{(i^K+\delta)}{\frac{\epsilon-1}{\epsilon}\alpha\delta} - 1} - \frac{\alpha}{1-\alpha} \frac{1+\varphi}{\varphi} \\ &\rightarrow \frac{\partial \ln C}{\partial i^K} < 0 \Leftrightarrow 1 + \frac{1-\alpha}{\alpha} \frac{\varphi}{1+\varphi} < \frac{i^K + \delta}{\frac{\epsilon-1}{\epsilon}\alpha\delta}, \end{aligned}$$

which is true if δ is sufficiently close to zero. ■

Next, consider the case of a **non-separable utility** function. For concreteness, assume consumption and money balances are complements. In this case, Y is not only a function of i^K (which is a function of the interest rate on reserves, i^M, as mentioned above) but is also a function of Q, which in turn is a function of the CBDC interest rate. As the interest rate on the CBDC rises, there are two effects. **Directly**, the opportunity cost of holding CBDC balances falls, so the "inflation" tax imposed on consumption declines and consumption increases. Moreover, leisure becomes less valuable, so the supply of labor increases and more output is produced. This result is summarized in Lemma 2.

Lemma 2 *Assume $\eta < \sigma$. Then, an increase in i^F increases output, capital and investment; also consumption increases.*

Proof. An increase in i^F decreases Q according to (9), which increases Y from (20). ■

There is an **indirect** effect of the CBDC interest rate as well, which operates in the transitional dynamics, not in the steady state though. An increase in the CBDC interest rate puts pressure on banks to increase the interest on deposits to the extent that CBDC and bank deposits are substitutes. This increases the cost of funding for banks, pushing up the cost of loans for firms. Therefore, the supply side of the economy is negatively affected. This channel does not operate in the steady state because the nominal interest rate is fixed, so the interest rates on deposits and loans are solely determined by the interest on reserves (as long as the leverage constraint is binding) and there is no transmission from the funding to lending sides of the banks.

Block 4, a binding leverage constraint gives M/P:

$$\frac{D}{P} = \ell \left(\frac{M}{P} + \rho b \right), \tag{21}$$

where $b = K$ comes from block 3 and $D/P = CV_D^{-1}$ comes from block 1 (for V_D) and block 3 (for C).

At the end, $w = \frac{\epsilon-1}{\epsilon}(1-\alpha)\frac{Y}{L}$ gives w, and $I = \delta K$. Also, $F/P = CV_F^{-1}$ gives F.

We cover some special cases in Appendix A.

4 Responses to Shocks

In this section and the next, we study the effects of various shocks to this economy by log-linearizing the model around the steady state. In Appendix A, we collect the equilibrium conditions in a benchmark model without a CBDC as well as equilibrium conditions of the main model with a CBDC. We do not repeat those equations here except for a few of them that provide insights new to the literature.

In terms of notation, the log-linearized version of X_t is denoted by $\widehat{x}_t$. Also, $\widetilde{x}_t \equiv \widehat{x}_t - \widehat{p}_t$ for $x \in \{d, f, m\}$ denotes the real balances for deposits, the CBDC and reserves. Parameters α_{FD}, β_J, α_m, α_c, α_y, and α_{DD} are all constant and defined in the appendix. Finally, we define $\widehat{\pi}_t \equiv \Delta\hat{p}_t$.

4.1 Solving the Model Off the Steady State

The unknown variables are as follows:

- Output, consumption and labor: $\widehat{y}_t, \widehat{c}_t, \widehat{l}_t$
- Output price inflation, real intermediate price, real wage, and variable Q: $\widehat{\pi}_t$, $\widehat{p}_{mt}, \widehat{w}_t$, $\widehat{q}_t$
- Real balances: $\widetilde{d}_t, \widetilde{m}_t, \widetilde{f}_t$
- Capital and investment: $\widehat{k}_t$ and $\widehat{i}_t$
- Real bank loans: $\widehat{b}_t$
- Rates: i_t^S, r_t^K, i_t^D, i_t^F, i_t^M

There are 15 equations describing the equilibrium optimality and market clearing conditions. See the log-linearized equations (49) to (63) in Appendix A. Given that there are 18 unknowns and 15 equations, 3 policy equations are needed to close the model. These equations should describe the policy variables regarding the rate or quantity of various types of liabilities issued by the central bank. The choice of these three variables is at the central bank's discretion, which is described below for different exercises. In the next section, we report the impulse responses comparing different exercises.

4.1.1 Benchmark Exercise: Traditional Monetary Policy Rule without a CBDC

In the benchmark exercise, we assume that a CBDC does not exist. Given that two unknowns (rate and quantity of CBDC) and one equation (CBDC demand) is removed, the central bank needs to set only 2 policy equations. We set a rule for interest on reserves and fix the quantity of reserves:

$$\begin{aligned} i_t^M &= r^M + \phi_\pi^M \Delta\hat{p}_t + \phi_y^M \hat{y}_t + u_t^M, && (22) \\ \widetilde{m}_t &= u_t^m. && (23) \end{aligned}$$

The terms u_t^M and u_t^m are shocks to the reserves interest rate and reserves quantity, respectively. We shock these variables one by one and report results in the next section. Also, the next few exercises all include CBDC in the model.

4.1.2 Exercise A1: Traditional Monetary Policy Rule

In the full model with a CBDC, we need to set 3 policy equations. We set a Taylor rule for reserves interest rate and fix the quantity of reserves and CBDC. Therefore, we use the same equations as in the benchmark exercise, (22) and (23), as well as the following:

$$\widetilde{f}_t = u_t^f, \tag{24}$$

where the term u_t^f denotes the shock to the CBDC quantity.

4.1.3 Exercise A2: CBDC Rule Instead of Reserves Rule

The monetary policy in the above economy works through the interest rate on reserves. In principle, we can investigate many other rules that the central bank can follow, like rules on the quantity of CBDC or reserves. In particular, we want to compare the implications of the benchmark economy with an economy in which CBDC is the main tool for MP. Therefore, we replace (22) with

$$i_t^F = r^F + \phi_\pi^F \Delta\hat{p}_t + \phi_y^F \hat{y}_t + u_t^F. \tag{25}$$

The other two equations to close the model are the same as those in A1, i.e., (23) and (24).

4.1.4 Exercise A3: Fixed-Interest CBDC

Many central banks consider only a zero-interest rate CDBC and do not plan to use the interest rate of a CBDC as an active monetary policy tool. Here, we use the same monetary policy rule as in A1, i.e., (22), but we assume that the interest rate on the CBDC is zero in the steady state but changes according to an exogenous process:

$$i_t^F = u_t^F, \tag{26}$$

where u_t^F is the shock to the interest rate on the CBDC. We now only need one equation to close the model. We use the same reserves quantity rule as in A1, (23).

4.2 Monetary Policy Transmission

Before presenting the results formally, we discuss at an abstract level how monetary policy transmission works in this model. The monetary policy here can work through the interest rate on reserves, the interest rate on CBDC, and the quantity of real balances of reserves and the CBDC.

To elaborate, combine (3) and bank optimality conditions to obtain

$$\underbrace{\frac{i_t^S - i_t^F}{1+i_t^S}}_{\text{CBDC spread}} - \underbrace{\left[\left(1 - \frac{\omega_D}{\omega_{FD}} \left(\frac{F_t}{D_t}\right)^{\frac{1}{\upsilon}}\right) \underbrace{\frac{i_t^S - i_t^F}{1+i_t^S}}_{\text{CBDC spread}} \right]}_{\text{Spread between CBDC and deposits}}$$

$$= \underbrace{\frac{i_t^S - i_t^D}{(1+i_t^S)}}_{\text{Deposits spread}} = \frac{1}{\ell} \underbrace{\frac{i_t^S - i_t^M}{(1+i_t^S)}}_{\text{Reserves spread}} = \frac{1}{\rho\ell} \underbrace{\frac{i_t^S - E_t i_{t+1}^K}{(1+i_t^S)}}_{\text{Capital return spread}} . \tag{27}$$

This equation is key for understanding monetary policy transmission in this paper.

First, let's take a special case of $\upsilon = \infty$ and $\omega_D = \omega_{FD}$. These assumptions represent the case where the CBDC and deposits are perfect substitutes. As a result, the interest on deposits and CBDC should be identical, otherwise one would not be used in equilibrium; so, $i_t^D = i_t^F$. We will then have

$$\frac{i_t^S - i_t^M}{(1+i_t^S)\,\ell} = \frac{i_t^S - i_t^F}{1+i_t^S} = \frac{i_t^S - E_t i_{t+1}^K}{(1+i_t^S)\,\rho\ell}.$$

This equation reveals that in the case of perfect substitution and binding reserve requirement, the transmission of MP through CBDC or reserves is identical. For any given i_t^M, one can find an i_t^F that keeps the demand for deposits and the bank FOCs unchanged. Precisely setting $i_t^F = \frac{i_t^M - (1-\ell) i_t^S}{\ell}$ replicates the same allocation as i_t^M. The rest of the equations in the model are unchanged.[11]

In the general case of (27) where CBDC and deposits are not perfect substitutes (where i^D and i^F are different), effects of i_t^M and i_t^F could be different because the spread (the large term in (27)) changes in a complicated way.

In the monetary policy regime based on interest on reserves, i_t^M, a change in the monetary policy interest rate changes the demand side of the economy by changing the rate on deposits. The deposit rate then changes the demand side of the economy through money demand, the Euler equation, and the labor supply. It also changes the supply side of the economy through the change in the rate of return on capital.

In the monetary policy regime based on the interest on CBDC, i_t^F, a change in the monetary policy interest rate changes the demand side of the economy through money demand, the Euler equation, and the labor supply. This also changes the interest on deposits. The change in the deposit rate then changes the rate of return on capital, which affects the supply side of the economy.

[11] In particular, note that Q remains the same given that $i_t^D = i_t^F$ and $\omega_D = \omega_{FD}$.

5 Quantitative Exercise

In this section, we first calibrate the model. We then study the impulse responses of the model to different shocks in the next subsections. In particular, we shock the interest rate on and quantity of reserves and the interest rate on CBDC.

5.1 Calibration

The calibration parameters are summarized in table 1.

Table 1: Calibration

Parameter	Value	Note	Parameter	Value	Note
β	1/1.04	utility discount factor	$\bar{A}$	1	steady-state TFP
ρ	0.4	collateral	ϵ	6	intermediate good demand elasticity
l	1/1.1	financial constraint	δ	0.05	capital depreciation rate
ω_D	0.5	utility weight of deposit	κ	75	price stickness
ω_{FD}	0.5	utility weight of deposit - CBDC	*cost*	3	investment adjustment cost
η	1.5	substitute between consumption and money	ϕ_π	1.5	Taylor rule, response to inflation
υ	0.8 or 2	substitute between deposit and CBDC	ϕ_Y	0	Taylor rule, response to output
σ	2	relative risk aversion	μ_{iM}	0.75	persistence: reserves interest rate
φ	1	Frisch elasticity of labor supply	μ_{iF}	0.75	persistence: CBDC interest rate
ψ	1	disutility of labor	μ_M	0.75	persistence: monetary supply
α	0.35	technology: capital share	μ_F	0.75	persistence: CBDC supply

For most parameters, we use the estimates in the literature.

For the zero inflation steady state, we have to set the following parameters. We set the real rate to be around 4%, so the nominal rate should be 4% as well given the zero inflation rate. The nominal rate on reserves is set to $i_M = 0.25\%$, which was the prevailing rate for several years before the COVID-19 pandemic. We also set the interest rate on the CBDC to $i_F = 0$ for the steady state. The values for elasticity of substitution between consumption and means of payments is set as $\sigma = 2$, and the inter-temporal elasticity of substitution between the consumption bundle today and tomorrow is set as $\eta = 1.5$, both following Piazzesi et al. (2019).

For the substitution between CBDC and deposits, of course, we don't have data. We set it to $\upsilon = 0.8$ or 2 to capture either the case where the CBDC and deposits are complements or where they are substitutes. The model is flexible enough and allows experimentation with different values of design parameters. For example, σ and υ need not be the same. The value for the rest of the parameters are standard in the literature.

5.2 Results without a CBDC

In this part, we first consider the case without a CBDC and study the reserves interest rate shock and the reserves quantity shock. The impulse response functions (IRFs) of the endogenous variables to the two shocks are shown in the following figures. In most cases, the directions of responses are as expected. With a positive shock to interest on reserves, output,

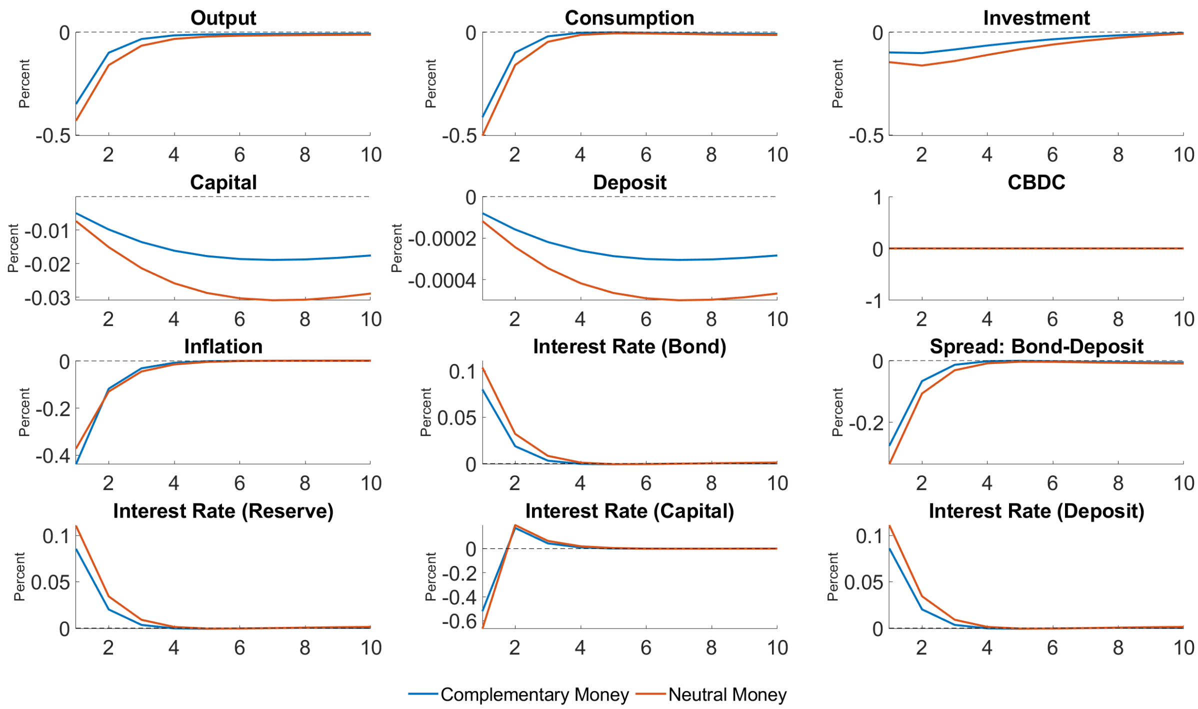

Figure 1: Results without a CBDC: Reserve Interest Rate Shock

consumption, and inflation all decrease. With a positive shock to quantity of reserves, all of them increase.

Since the banks are the only agents in this economy holding reserves, a change in the reserves rate affects only banks and the rest of the actions follow from banks' responses. According to the banks' optimality condition, (27), in response to the increase of the reserves interest rate, there is (i) upward pressure on the interest rate of the bond, (ii) downward pressure on the interest rate spread between the bond and the deposit, and (iii) upward pressure on the real return on capital.[12] These three channels interplay, illustrating the mechanisms through which a monetary policy shock influences economic variables.

First, as i_t^S increases, the traditional NK channel is triggered. An increase in the nominal bond interest rate (i_t^S) leads to adjustments in Euler's equation, resulting in reduced current consumption (C_t). This decrease in consumption dampens aggregate demand and hence aggregate output (Y_t), subsequently reducing inflation (π_t) through the NK Phillips curve.

Second, the NM channel comes into play, wherein the spread between the short-term nominal interest rate (i_t^S) and the deposit rate (i_t^D) affects the cost of liquidity. A narrower spread lowers the cost of liquidity, encouraging higher deposit demand and thus stimulating

[12]In our discussion here when we say that there is an upward (downward) pressure on a certain variable, we mean that an increase (decrease) in that variable is consistent with equilibrium conditions. For some variables, a change in another direction could also be consistent with equilibrium conditions from a theoretical perspective under a counterfactual calibration.

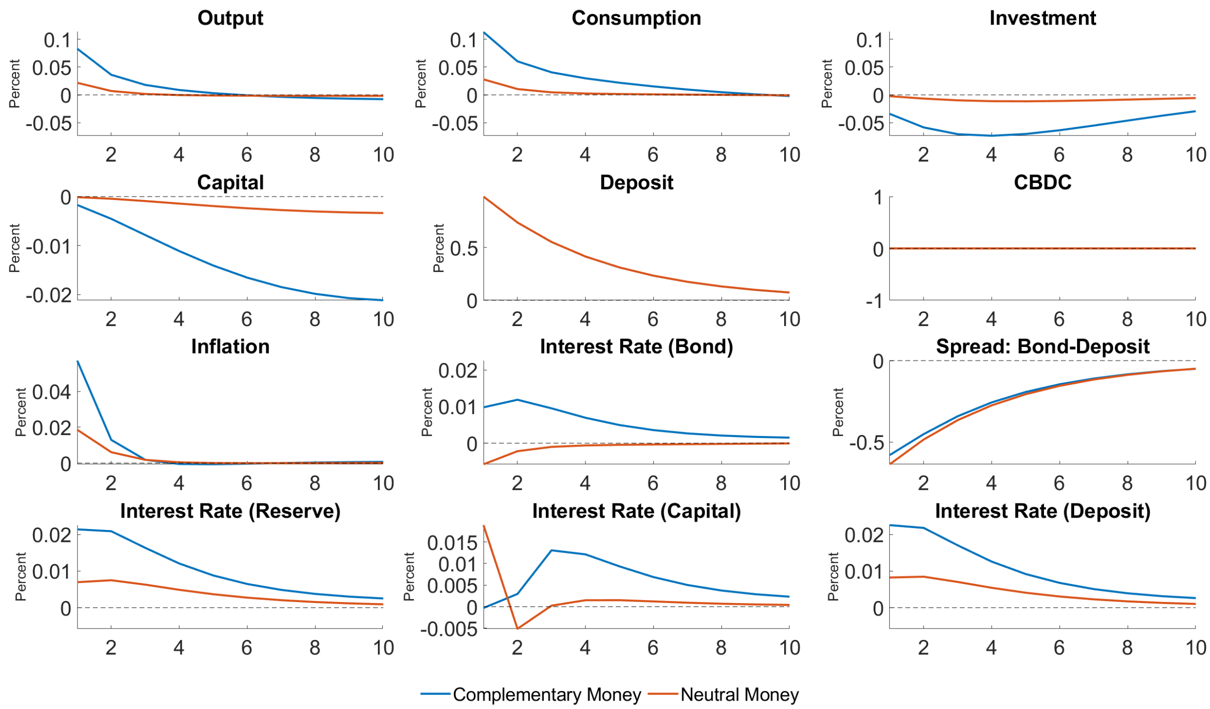

Figure 2: Results without a CBDC: Reserve Quantity Shock

consumption. This increase in consumption subsequently boosts labor supply and output levels, counteracting the initial dampening effect of the monetary policy shock. Figure 1 compares a scenario when money is a complement to consumption ($\sigma > \eta$) with one when money is neutral ($\sigma = \eta$), which indicates that the NM channel is shut down. This comparison elucidates the contribution of the NM channel to overall economic dynamics. As we can see from the graph and consistent with our explanation, the response of output and consumption will slightly increase when the NM channel is shut down.

Lastly, the supply channel plays a role whereby the higher marginal cost of capital in the following period ($t + 1$) discourages investment, leading to a decline in current output (Y_t). This explains the overall decline in investment along the transition path. There is also a sharp increase in the real rate of capital from the first to second period after the shock. This increase is due to the fact that the capital stock (K_t) is predetermined, with lower output (Y_t) resulting in a decrease in the profit per unit of capital and subsequently real rate of return on capital in the current period t (see (10)) but an increase in the subsequent period $t + 1$ due to the decrease in current investment.

The IRFs of a reserve quantity shock are reported in Figure 2. Compared with a reserve interest rate shock, a reserve quantity shock triggers a different series of adjustments. An increase in reserves supply relaxes the leverage constraint, leading to an expansion of deposits and a reduction in capital accumulation. That is, an expansion of reserves crowds out real investment opportunities in the model.

The reserves expansion leads to several consequential effects. Firstly, it contributes to a decrease in the bond-deposit spread, as depicted by the deposit demand equation. This reduction in the spread is consistent with lower bond returns, thus stimulating higher consumption, as indicated by the Euler equation. Additionally, the complementarity between money and consumption further amplifies this effect, resulting in a larger increase in consumption levels. Simultaneously, the decrease in capital accumulation leads to a decline in output, but the overall effect on aggregate output is still positive given the large response of comsumption.

The comparison between complementary money and neutral money in Figure 2 shows that despite the absence of complementarity between money and consumption, output and consumption both expand, revealing that the standard NK channel dominates the supply channel.

5.3 No CBDC vs. Zero-Interest CBDC

In this section, we compare the effects of shocks with or without a fixed-interest-rate CBDC, and the interest rate is assumed to be zero unless otherwise noted, as in Section 5.4. A fixed-interest-rate CBDC represents one of the simplest methods of introducing CBDC and is advocated by many policymakers. The shocks we consider here are a standard monetary policy shock (u_t^{im}) as well as a reserves quantity shock (u_t^m). We find that the **effects of CBDC depends on the type of shocks** hitting the economy. The response to an interest rate shock is **amplified** with a fixed-interest-rate CBDC. However, the response to a quantity shock is **dampened** with a fixed-interest-rate CBDC compared to the case with a CBDC.

5.3.1 Reserves Interest Rate Shock

The IRFs of a reserves interest rate are reported in Figure 3. Comparing the IRFs of the model incorporating CBDC with the baseline model lacking CBDC, as illustrated in Figure 1, reveals that **the introduction of a CBDC amplifies the impact of a contractionary traditional monetary policy shock**. This amplification remains irrespective of the elasticity of substitution between deposits and CBDC (υ). The difference between the response of investment is relatively small here because the behavior of interest rates is almost identical across these scenarios except for the illiquid bonds interest rates that are slightly different.

To better understand why the effects of a traditional monetary policy shock are amplified, we combine the log-linearized version of the demand functions for the CBDC and the deposit to obtain

$$\widetilde{d}_t - \widetilde{f}_t = \upsilon \left(\frac{i_t^S - i_t^F}{i^S - i^F} - \frac{i_t^S - i_t^D}{i^S - i^D} \right),$$

which suggests that the difference between the demands of different assets depends on the difference of the spreads. Using this equation, we can eliminate q_t and simplify the demand

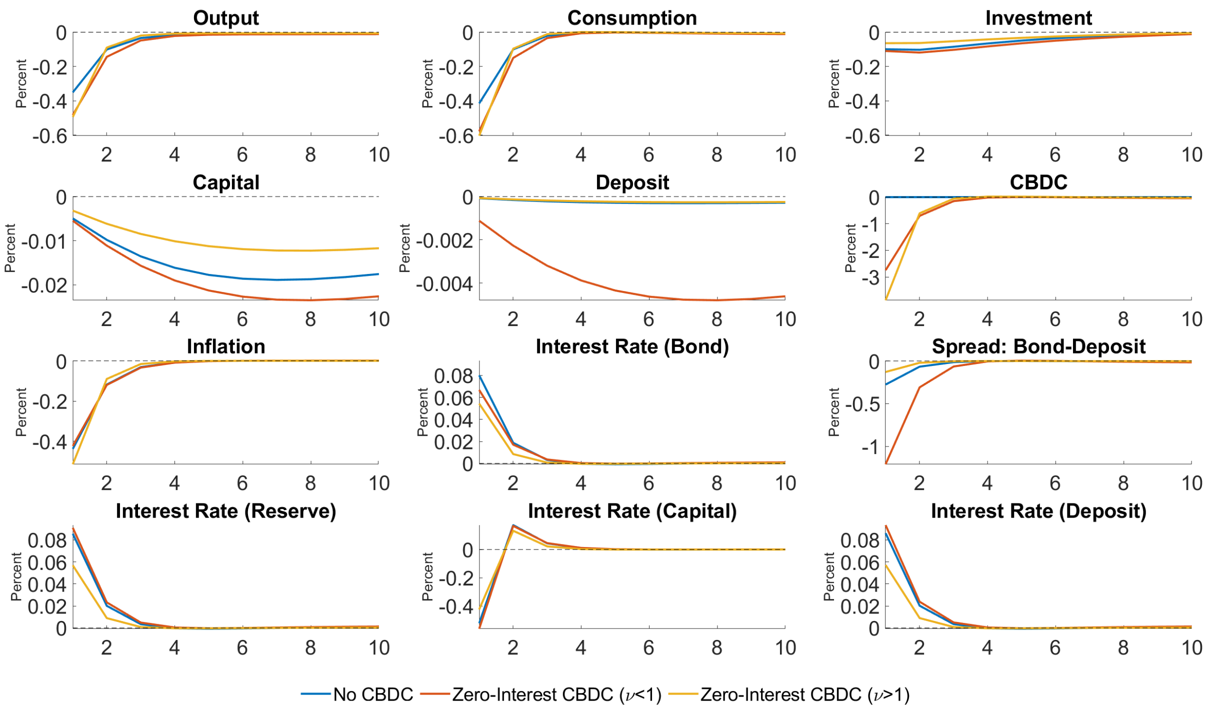

Figure 3: No CBDC vs. Zero-Interest CBDC: Reserves Interest Rate Shock

functions for the deposit and CBDC:

$$\frac{i_t^S - i_t^D}{i^S - i^D} - 1 - (1-\beta_D)\left(\frac{\upsilon}{\eta} - 1\right)\left(\frac{i_t^S - i_t^F}{i^S - i^F} - \frac{i_t^S - i_t^D}{i^S - i^D}\right) = \frac{1}{\eta}\widehat{c}_t - \frac{1}{\eta}\widetilde{d}_t \tag{28}$$

$$\frac{i_t^S - i_t^F}{i^S - i^F} - 1 + \beta_D\left(\frac{\upsilon}{\eta} - 1\right)\left(\frac{i_t^S - i_t^F}{i^S - i^F} - \frac{i_t^S - i_t^D}{i^S - i^D}\right) = \frac{1}{\eta}\widehat{c}_t - \frac{1}{\eta}\widetilde{f}_t. \tag{29}$$

We use these two equations to eliminate the quantity of the deposit and CBDC in the definition of Q to obtain

$$\widehat{q}_t = \alpha_{DD}\left(\beta_D\left(\frac{i_t^S - i_t^D}{i^S - i^D} - 1\right) + (1-\beta_D)\left(\frac{i_t^S - i_t^F}{i^S - i^F} - 1\right)\right), \tag{30}$$

where α_{DD} and β_D are parameters and β_D depends on the steady-state fraction of the deposit in the composition of Q. Equation (30) is essential to the analysis of an increase in the reserve interest rate shock. In the baseline model in the absence of a CBDC, $\beta_D = 1$ and, hence, the equation will only depend on the spread between i^S and i^D. When the reserves interest rate increases, again using the bank optimality condition, the bond interest rate tends to increase and the spread between i^S and i^D tends to decrease. In the absence of a CBDC, the decrease in $i^S - i^D$ suggests that Q will decrease. This is the NM channel discussed earlier that can reduce the decrease in total output.

However, with a zero-interest-rate CBDC, i_t^F is zero and, hence, $i^S - i^F$ will increase. Given that β_D always lies between 0 and 1, the introduction of a CBDC diminishes the impact

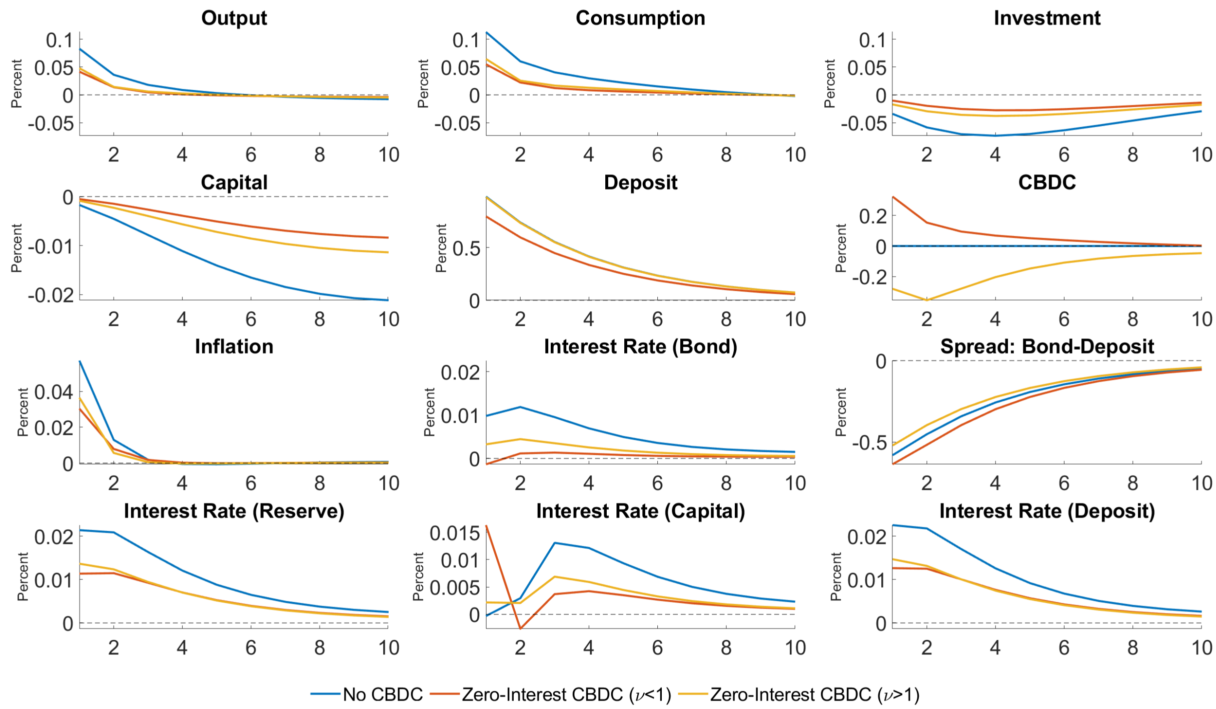

Figure 4: No CBDC vs. a Zero-Interest CBDC: Reserves Quantity Shock

stemming from the decrease in the spread between i^S and i^D. Consequently, the NM channel is attenuated. Since the NM channel serves to mitigate the decline in output, its attenuation amplifies the decrease in output.

5.3.2 Reserves Quantity Shock

The IRFs of a reserves quantity shock are reported in Figure 4. Conventional wisdom would suggest that the reserves interest rate should fall following an increase in the quantity of reserves. However, in this model, a higher quantity of reserves means that banks' leverage constraint becomes less tight, so they can issue more deposits, for which banks are required to pay a higher interest rate. Banks' optimality conditions then imply that the interest on reserves should rise as well. Since interest on deposits rises, consumption and labor supply both increase significantly because of complementarity between consumption and money balances. Given that consumption increases significantly but output does not increase as much, investment should fall. Changes in the investment are supported by a higher rate of return on capital for some periods.

Comparing the effects of the quantity shock across the two scenarios, with and without a CBDC, reveals that the response of consumption, output, and inflation is **weaker** in the case with a CBDC almost irrespective of the design of the CBDC. Interesting, without a CBDC, investment declines considerably in the first few periods and starts recovering only after that. With a CBDC, the decline is much smaller. CBDC, even though not interest bearing, dampens the response of investment and subsequently consumption and output.

This dampened response seems to be implied by a modest increase in the interest rate of reserves and deposits in the case with a CBDC.

When CBDC and deposits are better complements ($\upsilon < 1$), their quantities move in the same direction. When they are better substitutes ($\upsilon > 1$), their quantities move in opposite directions. We do not observe this in the reserves interest rate shock case, mainly because the change in i^S is large enough to dominate the effects arising from substitution or complementary between the CBDC and the deposit. Unlike in the reserves interest rate shock, the reserves quantity shock leads to an expansion in deposit supply, thereby reducing the opportunity cost of holding deposits. This will make the change in i^S much smaller in magnitude.

In order to explain the milder effects on output and consumption in the presence of a CBDC, we again turn to equation (28). We compare it to the same log-linearized deposit demand function in the baseline model but in the absence of a CBDC:

$$\frac{i_t^S - i_t^D}{i^S - i^D} - 1 = \frac{1}{\eta}\widehat{c}_t - \frac{1}{\eta}\widetilde{d}_t.$$

We observe that the right-hand side of the two equations are the same. Once a CBDC is added, the demand for deposits is not only affected by the spread on bank deposits but also by the spread on the CBDC. In response to the change in the reserves quantity, the banks will increase the supply of deposits, d. In the baseline model, this will decrease the opportunity cost of holding deposits, $i^S - i^D$. In the model with a CBDC, the increase in the overall demand for deposits can be accommodated by the decrease in the opportunity cost of holding deposits and/or opportunity cost of holding CBDC. The smaller change in $i^S - i^F$ consequently weakens the NM channel. An attenuated NM channel dampens the effect of the shock on output and consumption.

5.4 CBDC vs. Reserves as the Main Monetary Policy Tool

Here, we compare the dynamics of two different policy scenarios. In the first case, the central bank uses the interest rate on CBDC as the main monetary policy tool, and the quantities of the reserves and CBDC are fixed. In the second case, the central bank uses the reserves interest rate as the main policy tool, but the interest rate of CBDC follows a simple pre-determined AR(1) process. See (26). We compare these two cases in terms of the response to a positive CBDC interest rate shock. The results are reported in Figure 5. The figure indicates that different policy rules will produce different and, in some cases, opposite responses.

The main point in this exercise is that the main monetary policy tool that is used in response to a CBDC interest rate shock matters significantly. If reserves are used as the main monetary policy tool (red curves), the CBDC interest rate shock is expansionary. CBDC interest rate payments reduce the opportunity cost of money for households, encouraging higher labor supply and more consumption, the main mechanism that exists in NM models. The NM channel plays an important role in this case, wherein the cost dynamics of money influence economic behavior, ultimately shaping overall economic outcomes. Also, it is interesting to

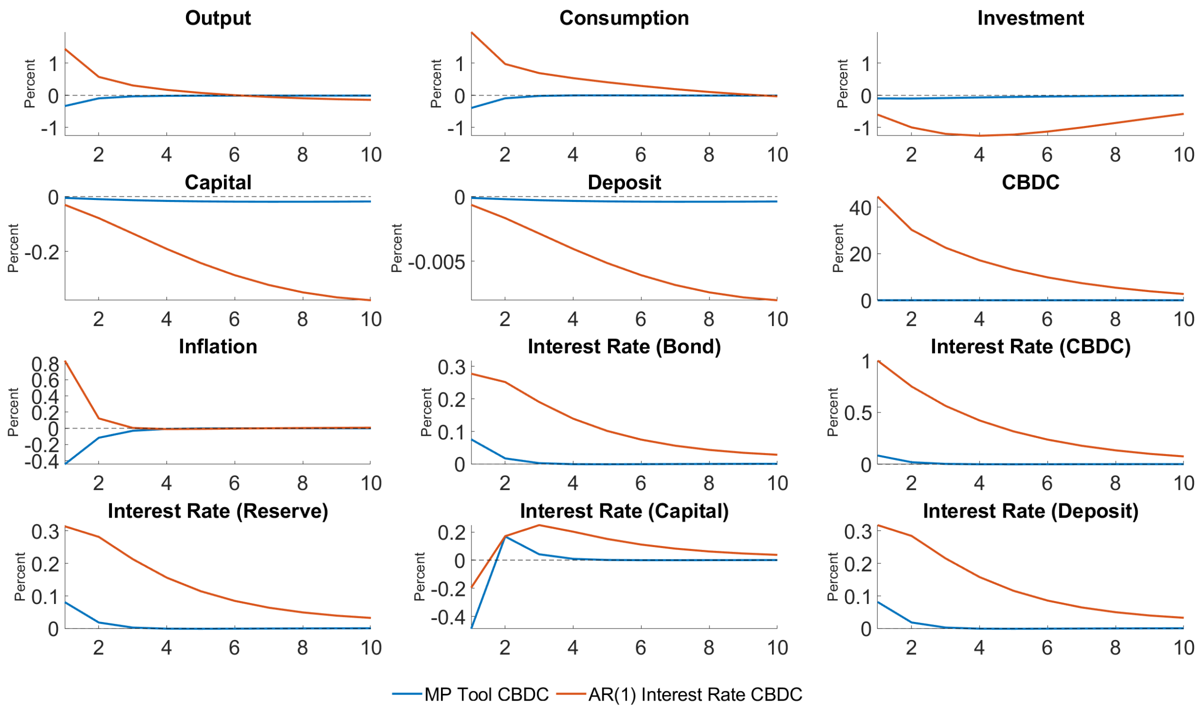

Figure 5: Monetary Policy Tool: CBDC vs. Reserves

note here that, although output and consumption both expand, investment and consequently capital reduce significantly. Intuitively, CBDC offers an interesting investment opportunity, but this crowds out real investment in this economy.

When CBDC is utilized as the main tool of monetary policy (blue curves), an increase in the CBDC interest rate is contractionary, akin to that observed with traditional reserves in most NK models.

5.5 Balance Sheet Quantity Rule Matters

In this section, we demonstrate that the reaction to a standard monetary policy shock depends on whether the central bank fixes the quantity of CBDC. In other words, the quantity rule of CBDC matters. To illustrate this point, we consider two scenarios: one where the central bank fixes the quantity of CBDC and the CBDC interest rate adjusts endogenously; and another where the central bank fixes the CBDC interest rate and the quantity of CBDC adjusts endogenously. Figure 6 reports the results.

The policy rule that fixes the CBDC interest rate yields a significant decline in output and consumption compared with the case of the central bank fixing the CBDC quantity. A positive shock to the reserves interest rate puts pressure on illiquid bonds and deposit interest rates in both cases. When the CBDC rate is flexible, the interest rate on CBDC increases as well, given that CBDC and deposits are imperfect substitutes, activating the NM channel and partially mitigating the contractionary effects on consumption and output. When the

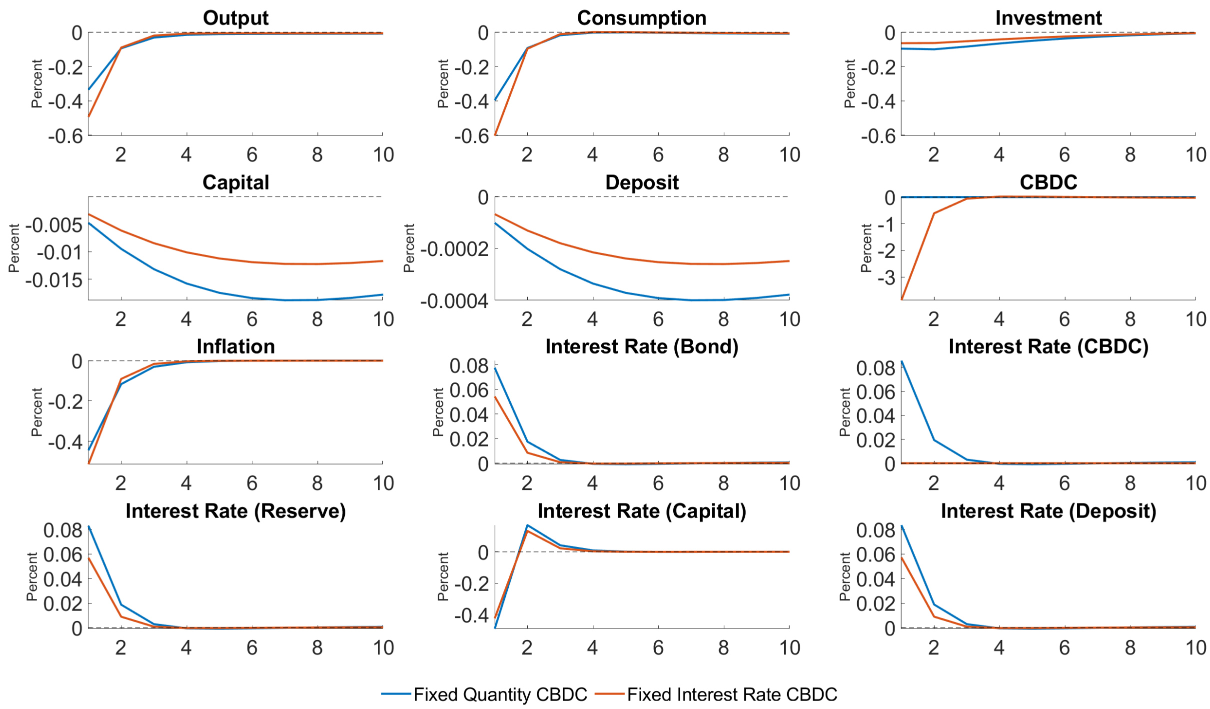

Figure 6: Fixed Quantity of CBDC vs. Fixed Interest Rate of CBDC

interest rate on CDBC is fixed, the opportunity cost of holding CBDC rises following a rise in the illiquid bonds interest rate. Given the higher opportunity cost of CBDC, the liquidity becomes effectively more expensive, leading to a lower level of consumption and output. Moreover, the quantity of CBDC adjusts downwards according to the money demand equation for it.

This exercise illustrates that in a world with CBDC, the quantity rule with respect to CBDC matters for determining the response of the economy to a standard monetary policy shock. This is in sharp contrast to standard NK models in which quantity of money is irrelevant for real variables.

6 Conclusion

Our paper explores the transmission of monetary policy shocks with or without a CBDC using a New Keynesian framework that incorporates financial frictions and a novel liquidity mechanism. Our analysis reveals that a CBDC can amplify or mitigate the effects of shocks, depending on the nature of the shock and the monetary policy framework. Moreover, different choices on design features of the CBDC can give rise to different implications on investment and other variables.

In response to a traditional monetary policy shock, the introduction of a zero-interest CBDC tends to magnify the contractionary effects. This occurs because the presence of a zero-interest CBDC raises the overall cost of liquidity, weakening the New Monetarist channel that typically offsets some of the downturn in output and consumption. Conversely, in the case of a positive reserve quantity shock, our results indicate that a CBDC could mitigate the effects on investment, consumption, and output.

Furthermore, our paper underscores the importance of the monetary policy framework adopted by a central bank. Different approaches to setting the interest rate, whether through a CBDC or, traditionally, through reserves, can lead to markedly different outcomes. When the CBDC interest rate is used as the primary tool of monetary policy, the economy responds to the CBDC interest rate shock in the same manner as it responds to a traditional monetary policy shock in the standard monetary policy framework. However, when reserves are used as the main tool, the response to a CBDC interest rate shock can be expansionary, enhancing labor supply and consumption through mechanisms akin to the NM channel.

Our framework can be used flexibly to study other questions that we did not address in this paper. For example, in the appendix we have added cash to the model, which naturally brings about an effective lower bound for nominal rates in this model. One could use this extended version of the model to study the effects of supply shocks or shocks to banks' financial conditions. Another research topic is the introduction of quantitative easing in this framework and comparing its effects with those of a CBDC, as they look like similar measures in some papers (such as Barrdear and Kumhof (2022)). All these topics are left for future research.

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Appendix

A Equilibrium conditions in other cases

This appendix consists of several sections.

A.1 Equilibrium conditions for the benchmark model (without a CBDC and cash)

$$\text{Illiquid bond demand:} \quad \beta E_t \left[\left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{\eta}{\sigma}-1} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\eta}} \frac{P_t}{P_{t+1}} \right] \left(1 + i_t^S\right) = 1$$

$$\text{Deposit demand:} \quad \frac{i_t^S - i_t^D}{1 + i_t^S} = \omega_D \left(\frac{P_t C_t}{D_t} \right)^{\frac{1}{\eta}}$$

$$\text{Bank FOCs:} \ \frac{i_t^S - i_t^D}{1 + i_t^S} = \frac{i_t^S - i_t^M}{\left(1 + i_t^S\right) \ell} = \frac{i_t^S - E_t i_{t+1}^K}{\left(1 + i_t^S\right) \rho \ell}$$

$$\text{Production and market clearing: } Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} = C_t + I_t + \frac{\kappa}{2} \left(\frac{P_t}{P_{t-1}} - 1 \right)^2$$

$$K_{t+1} = b_t = I_t + (1 - \delta) K_t$$

$$\text{Labor demand:} \quad w_t = p_{mt} (1 - \alpha) \frac{Y_t}{L_t}$$

$$\text{Capital demand:} \quad 1 + r_t^K = \alpha p_{mt} \frac{Y_t}{K_t} + 1 - \delta$$

$$\text{Labor supply:} \quad w_t = Q_t^{1-\frac{\eta}{\sigma}} C_t^{\frac{1}{\sigma}} \psi L_t^{\varphi}$$

$$\rightarrow p_{mt} Y_t = \frac{\left(\delta + r_t^K\right) K_t}{\alpha} = Q_t^{1-\frac{\eta}{\sigma}} C_t^{\frac{1}{\sigma}} \psi \frac{L_t^{1+\varphi}}{(1 - \alpha)}$$

$$\text{Optimal pricing:} \ \left[\frac{\epsilon - 1}{\epsilon} - p_{mt} \right] \frac{\epsilon Y_t}{\kappa} + \frac{P_t}{P_{t-1}} \left(\frac{P_t}{P_{t-1}} - 1 \right) = E_t \left[\Lambda_{t,t+1} \frac{P_{t+1}}{P_t} \left(\frac{P_{t+1}}{P_t} - 1 \right) \right]$$

$$Q_t \equiv \left(1 + \omega_D V_{D,t}^{-(1-\frac{1}{\eta})} \right)^{\frac{1}{1-\eta}}$$

$$\text{Leverage constraint } \frac{M_t}{P_t} = \frac{1}{\ell}\frac{D_t}{P_t} - \rho b_t$$

Unknowns:

- C, Y, K, L, I, Q
- $p_{mt}, \frac{P_{t+1}}{P_t}$
- $b, \frac{D_t}{P_t}, \frac{M_t}{P_t}$
- i_t^S, i_t^D, i_t^K

These are 13 equations and 14 unknowns, so we need an equation to close the model:

$$\text{Market clearing for reserves: } \frac{M_t^S}{P_t} = \frac{M_t}{P_t}.$$

Exogenous: $i_t^M, \frac{M_t^S}{P_t}$. [13] [14]

Steady State Conditions:

$$\begin{aligned}
\text{Intermediate good price:} \quad & p_m = \frac{\epsilon - 1}{\epsilon}, \\
\text{Illiquid bond demand:} \quad & \beta\left(1 + i^S\right) = 1.
\end{aligned}$$

Block 1: Given i^M, we can solve for i^K and i^D:

$$\frac{i^S - i^D}{1 + i^S} = \frac{i^S - i^M}{(1 + i^S)\,\ell} = \frac{i^S - i^K}{(1 + i^S)\,\rho\ell}. \tag{31}$$

Block 2: Given i^D from block 1, we can pin down $V_D \equiv \frac{PC}{D}$:

$$\text{Deposit demand: } \frac{i^S - i^D}{1 + i^S} = \omega_D V_D^{\frac{1}{\eta}}.$$

Moreover, we have:

$$b = K = \frac{I}{\delta}.$$

[13] Note that the leverage constraint must be binding all the time. If the constraint is slack, we must have $i_t^S = i_t^D = i_t^M = E_t i_{t+1}^K$; but in that case, the deposit demand equation implies that D should be very large, which means that the constraint cannot be slack!

[14] If the central bank sets the **nominal** value of reserves, then we have the following unknowns: $C, Y, K, L, I; p_{mt}, P_t; b, D_t, M_t; i_t^S, i_t^D, i_t^K$.The market clearing for reserves remains the same, $M_t^S = M_t$.

Block 3: Given i^K (from block 1) and $Q \equiv \left(1+\omega_D V_D^{-(1-\frac{1}{\eta})}\right)^{\frac{1}{1-\eta}}$ (from block 2), the following four equations pin down Y, L, C and b :

$$\begin{aligned} Y &= C + \delta K, \\ p_m &= \frac{\epsilon - 1}{\epsilon} = \frac{\left(i^K + \delta\right) K}{\alpha Y} \to K = \frac{\epsilon - 1}{\epsilon} \frac{\alpha Y}{i^K + \delta}, \\ Y &= A K^{\alpha} L^{1-\alpha}, \\ \frac{\epsilon - 1}{\epsilon} Y &= Q^{1-\frac{\eta}{\sigma}} C^{\frac{1}{\sigma}} \psi \frac{L^{1+\varphi}}{1-\alpha}. \end{aligned} \tag{32}$$

We can now calculate Y as a function of C and K, and then use the market clearing condition:

$$Y^{\frac{1+\varphi}{1-\alpha}-1} = \frac{\epsilon - 1}{\epsilon} \frac{(1-\alpha) A^{\frac{1+\varphi}{1-\alpha}} K^{\alpha \frac{1+\varphi}{1-\alpha}}}{Q^{1-\frac{\eta}{\sigma}} C^{\frac{1}{\sigma}} \psi},$$

$$Y = C + \delta K \to C = Y \left(1 - \frac{\epsilon - 1}{\epsilon} \frac{\alpha \delta}{i^K + \delta}\right). \tag{33}$$

Given that K and C are now given in terms of Y from the last two numbered equations, we obtain

$$Y^{\varphi + \frac{1}{\sigma}} = \frac{\alpha^{\frac{\alpha(1+\varphi)}{1-\alpha}} (1-\alpha) A^{\frac{1+\varphi}{1-\alpha}} \left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}-1}}{\psi Q^{1-\frac{\eta}{\sigma}} \left(1 - \frac{\epsilon-1}{\epsilon} \frac{\alpha\delta}{i^K+\delta}\right)^{\frac{1}{\sigma}} \left(i^K + \delta\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}}}. \tag{34}$$

Log-Linearized Version:

$$\text{Euler equation: } \widehat{c}_t = E_t\left[\widehat{c}_{t+1}\right] - \sigma\left(\beta i_t^S - E_t\left[\widehat{\pi}_{t+1}\right] + \beta - 1\right) + (\sigma - \eta)\left(E_t\left[\widehat{q}_{t+1}\right] - \widehat{q}_t\right) \tag{35}$$

$$\text{Deposit demand:} \frac{i_t^S - i_t^D}{i^S - i^D} - 1 = \frac{1}{\eta}\widehat{c}_t - \frac{1}{\eta}\widetilde{d}_t \tag{36}$$

Bank equations:

$$\text{Bank FOC:} \quad i_t^S - i_t^D = \ell^{-1}\left(i_t^S - i_t^M\right) \tag{37}$$

$$\text{Bank FOC} \; : \; i_t^S - E_t r_{t+1}^K - \left(1 + r^K\right) E_t\left[\widehat{\pi}_{t+1}\right] = \rho\left(i_t^S - i_t^M\right) \tag{38}$$

$$\text{Bank Leverage} \; : \; \widetilde{d} = \alpha_m \widetilde{m} + (1 - \alpha_m)\widehat{b}_t \tag{39}$$

Philips curve:

$$\widehat{\pi}_t = \frac{(\epsilon - 1) Y}{\kappa} \widehat{p}_{mt} + \beta E_t\left[\widehat{\pi}_{t+1}\right] \tag{40}$$

The rest of the equations:

$$\begin{aligned}
\widehat{y}_t &= \alpha\widehat{k}_t + (1-\alpha)\,\widehat{l}_t && (41)\\
\widehat{y}_t &= \alpha_c\widehat{c}_t + (1-\alpha_c)\,\widehat{i}_t && (42)\\
\widehat{k}_{t+1} &= \widehat{b}_t && (43)\\
\widehat{k}_{t+1} &= \delta\widehat{i}_t + (1-\delta)\,\widehat{k}_t && (44)\\
\widehat{w}_t &= \widehat{p}_{mt} + \widehat{y}_t - \widehat{l}_t && (45)\\
\frac{r_t^K - r^K}{1+r^K} &= \alpha_y\left(\widehat{p}_{mt} + \widehat{y}_t - \widehat{k}_t\right) && (46)\\
\widehat{w}_t &= \left(1-\frac{\eta}{\sigma}\right)\widehat{q}_t + \frac{1}{\sigma}\widehat{c}_t + \varphi\widehat{l}_t && (47)
\end{aligned}$$

$$\widehat{q}_t = \frac{1}{\eta}\alpha_{DD}\widehat{V}_{D,t} = \frac{1}{\eta}\alpha_{DD}\left(\widehat{c}_t - \widetilde{d}_t\right),$$

where

$$\alpha_{DD} \equiv \frac{\omega_D V_D^{\frac{1}{\eta}-1}}{1+\omega_D V_D^{\frac{1}{\eta}-1}}.$$

$$\text{Reserves interest rate (Taylor) rule}: i_t^M = r^M + \phi_\pi^M \Delta\hat{p}_t + \phi_y^M \hat{y}_t + u_t^M \qquad (48)$$

$$\text{Reserves quantity rule}: \widetilde{m} = u_t^{\text{Reserves}}$$

A.2 Problem with a CBDC

Using the following definitions:

$$\begin{aligned}
\widetilde{x} &= \widehat{x}_t - \widehat{p}_t \text{ for } x \in \{d, f, m\},\\
\alpha_{FD} &\equiv 1,\\
\beta_D &\equiv \frac{V_D^{-(1-\frac{1}{\upsilon})}}{V_D^{-(1-\frac{1}{\upsilon})} + \frac{\omega_{FD}}{\omega_D}V_F^{-\left(1-\frac{1}{\upsilon}\right)}},\\
\alpha_m &\equiv \frac{M/P}{M/P+\rho b},\\
\alpha_c &\equiv \frac{C}{Y},\\
\alpha_y &\equiv \frac{\alpha\frac{\epsilon-1}{\epsilon}\frac{Y}{K}}{\alpha\frac{\epsilon-1}{\epsilon}\frac{Y}{K} + 1 - \delta},\\
\alpha_{DD} &\equiv \frac{\omega_D V_{FD}^{\frac{1}{\eta}-1}}{1+\omega_D V_{FD}^{\frac{1}{\eta}-1}}.
\end{aligned}$$

Here is a summary of the log-linearized version of equilibrium conditions:

$$\text{Euler equation: } \widehat{c}_t = E_t\left[\widehat{c}_{t+1}\right] - \sigma\left(\beta i_t^S - E_t\left[\widehat{\pi}_{t+1}\right] + \beta - 1\right) + (\sigma - \eta)\left(E_t\left[\widehat{q}_{t+1}\right] - \widehat{q}_t\right) \quad (49)$$

$$\text{CBDC demand:} \quad \tfrac{i_t^S - i_t^F}{i^S - i^F} - 1 = \tfrac{1}{\eta}\widehat{c}_t - \left(\tfrac{1}{\eta} - \tfrac{1}{\upsilon}\right)\beta_D \widetilde{d} - \left(\tfrac{1-\beta_D}{\eta} + \tfrac{\beta_D}{\upsilon}\right)\widetilde{f} \quad (50)$$

$$\text{Deposit demand:} \quad \tfrac{i_t^S - i_t^D}{i^S - i^D} - 1 = \tfrac{1}{\eta}\widehat{c}_t - \left(\tfrac{1-\beta_D}{\upsilon} + \tfrac{\beta_D}{\eta}\right)\widetilde{d} - \left(-\tfrac{1}{\upsilon} + \tfrac{1}{\eta}\right)(1-\beta_D)\widetilde{f} \quad (51)$$

$$\text{Bank FOC:} \quad i_t^S - i_t^D = \ell^{-1}\left(i_t^S - i_t^M\right) \quad (52)$$

$$\text{Bank FOC:} \quad i_t^S - E_t r_{t+1}^K - \left(1 + r^K\right) E_t\left[\widehat{\pi}_{t+1}\right] = \rho\left(i_t^S - i_t^M\right) \quad (53)$$

$$\text{Bank Leverage:} \quad \widetilde{d} = \alpha_m \widetilde{m} + (1 - \alpha_m)\widehat{b}_t \quad (54)$$

$$\text{Philips curve: } \widehat{\pi}_t = \frac{(\epsilon - 1) Y}{\kappa}\widehat{p}_{mt} + \beta E_t\left[\widehat{\pi}_{t+1}\right] \quad (55)$$

The rest of equations:

$$\widehat{y}_t = \alpha \widehat{k}_t + (1 - \alpha)\widehat{l}_t \quad (56)$$

$$\widehat{y}_t = \alpha_c \widehat{c}_t + (1 - \alpha_c)\widehat{i}_t \quad (57)$$

$$\widehat{k}_{t+1} = \widehat{b}_t \quad (58)$$

$$\widehat{k}_{t+1} = \delta \widehat{i}_t + (1 - \delta)\widehat{k}_t \quad (59)$$

$$\widehat{w}_t = \widehat{p}_{mt} + \widehat{y}_t - \widehat{l}_t \quad (60)$$

$$\frac{r_t^K - r^K}{1 + r^K} = \alpha_y\left(\widehat{p}_{mt} + \widehat{y}_t - \widehat{k}_t\right) \quad (61)$$

$$\widehat{w}_t = \left(1 - \frac{\eta}{\sigma}\right)\widehat{q}_t + \frac{1}{\sigma}\widehat{c}_t + \varphi \widehat{l}_t \quad (62)$$

$$\widehat{q}_t = \frac{1}{\eta}\alpha_{DD}\left(\widehat{c}_t - \left(\beta_D \widetilde{d} + (1 - \beta_D)\widetilde{f}_t\right)\right) \quad (63)$$

Here are the exercises we plan to do.

Exercise A1:

$$i_t^M = r^M + \phi_\pi^M \Delta\hat{p}_t + \phi_y^M \hat{y}_t + u_t^M. \tag{64}$$

We need 2 more equations to close the model:

$$\widetilde{f}_t = u_t^f. \tag{65}$$
$$\widetilde{m}_t = u_t^m. \tag{66}$$

We can shock these variables, i_t^M, $\widetilde{f}_t$, $\widetilde{m}_t$ one by one.

Exercise A2:

$$i_t^F = r^F + \phi_\pi^F \Delta\hat{p}_t + \phi_y^F \hat{y}_t + u_t^F. \tag{67}$$

We need 2 more equations to close the model:

$$\widetilde{f}_t = u_t^f. \tag{68}$$
$$\widetilde{m}_t = u_t^m. \tag{69}$$

We can shock these variables, i_t^F, $\widetilde{f}_t$, $\widetilde{m}_t$ one by one.

Steady State Equations when the CBDC and deposits are perfect substitutes

$\upsilon = \infty$ and $\omega_D = \omega_{FD}$.

Steady State Equations:

- Output, consumption and labor: Y, C, L
- Deposits, CBDC and reserves balances: D, F, M
- Real assets: b
- Rates: i^K, i^D

Note that the nominal and real interest rates are equal because the inflation rate is zero, i.e., $i^K = r^K$ and $i^D = r^D$. We now derive the steady state values:

$$\text{Intermediate good price:} \quad p_m = \frac{\epsilon - 1}{\epsilon} ,$$
$$\text{Illiquid bond demand:} \quad \beta\left(1 + i^S\right) = 1.$$

Given the perfect substitution assumption, we have

$$\text{Perfect substitution:} \qquad i^D = i^F.$$

Note that i^M and i^F cannot be two independent policy tools. Rather, one can be calculated from the other. Here, we assume i^M is given.

Block 1: Given i^M, we can solve for i^K and $i^D = i^F$:

$$\frac{i^S - i^D}{1 + i^S} = \frac{i^S - i^M}{(1 + i^S)\,\ell} = \frac{i^S - i^K}{(1 + i^S)\,\rho\ell}. \tag{70}$$

Block 2: Given i^D from block 1 and i^F from policy, we can pin down V_D and V_F:

$$\text{Deposit and CBDC demand:} \qquad \frac{i^S - i^D}{1 + i^S} = \omega_D Q_D^{-\frac{1}{\eta}}. \tag{71}$$

Given $Q_D = \frac{D+F}{PC}$, we can calculate $Q \equiv \left(1 + \omega_D Q_D^{1-\frac{1}{\eta}}\right)^{\frac{1}{1-\eta}}$.

Moreover, we have

$$b = K = \frac{I}{\delta}.$$

Block 3: Given i^K (from block 1) and Q (from block 2), the following four equations pin down Y, L, C and b:

$$\begin{aligned}
Y &= C + \delta K, \\
p_m &= \frac{\epsilon - 1}{\epsilon} = \frac{\left(i^K + \delta\right) K}{\alpha Y} \to K = \frac{\epsilon - 1}{\epsilon} \frac{\alpha Y}{i^K + \delta}, \\
Y &= A K^\alpha L^{1-\alpha}, \\
\frac{\epsilon - 1}{\epsilon} Y &= Q^{1-\frac{\eta}{\sigma}} C^{\frac{1}{\sigma}} \psi \frac{L^{1+\varphi}}{1 - \alpha}.
\end{aligned} \tag{72}$$

We can now calculate Y as a function of C and K and then use the market clearing condition:

$$Y^{\frac{1+\varphi}{1-\alpha} - 1} = \frac{\epsilon - 1}{\epsilon} \frac{(1-\alpha) A^{\frac{1+\varphi}{1-\alpha}} K^{\alpha\frac{1+\varphi}{1-\alpha}}}{Q^{1-\frac{\eta}{\sigma}} C^{\frac{1}{\sigma}} \psi},$$

$$Y = C + \delta K \to C = Y\left(1 - \frac{\epsilon - 1}{\epsilon} \frac{\alpha\delta}{i^K + \delta}\right). \tag{73}$$

Given that K and C are now given in terms of Y from (72) and (19), we obtain

$$Y^{\varphi + \frac{1}{\sigma}} = \frac{\alpha^{\frac{\alpha(1+\varphi)}{1-\alpha}} (1-\alpha) A^{\frac{1+\varphi}{1-\alpha}} \left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{\alpha(1+\varphi)}{1-\alpha} - 1}}{\psi Q^{1-\frac{\eta}{\sigma}} \left(1 - \frac{\epsilon-1}{\epsilon} \frac{\alpha\delta}{i^K+\delta}\right)^{\frac{1}{\sigma}} \left(i^K + \delta\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}}}. \tag{74}$$

$$Q_D \equiv V_D^{-1} + V_F^{-1} = \frac{D+F}{PC}$$
$$Q \equiv \left(1 + \omega_D V_{FD}^{\frac{1}{\eta}-1}\right)^{\frac{1}{1-\eta}}.$$

Block 4, a binding leverage constraint gives M/P:

$$\frac{D}{P} = \ell\left(\frac{M}{P} + \rho b\right), \tag{75}$$

where $b = K$ comes from block 3 and $(D+F)/P = CQ_D$ comes from block 1 (for V_{FD}) and from block 3 (for C).

At the end, $w = \frac{\epsilon-1}{\epsilon}(1-\alpha)\frac{Y}{L}$ gives w, and $I = \delta K$.

IMPORTANT: We need to know either M/P or F/P to pin down the other quantities.

The log-linearized version of equilibrium conditions is no different than the model with general υ. We just need to set υ to a sufficiently large number.

B Role of Different Model Ingredients

B.1 The Role of Illiquid Bonds

We look at households and banks separately.

What if HHs cannot hold illiquid bonds?

We can define interest on illiquid bonds via the following equation:

$$\lambda_t = \beta \boldsymbol{E}\left[\lambda_{t+1}\left(1+i_t^S\right)\right]$$

If we define it this way, the equations should not be changed for households.

In this case, i_t^S can be calculated as follows:

$$1+i_t^S = \frac{\lambda_t}{\beta \boldsymbol{E}\lambda_{t+1}} = \frac{U_{C,t}}{\beta P_t \boldsymbol{E}\frac{U_{C,t+1}}{P_{t+1}}}.$$

This shows that the FOCs for households are unchanged compared with the case where households can hold illiquid bonds.

What if banks cannot hold illiquid bonds?

If banks cannot hold illiquid bonds, their problem can be written as

$$E_t\left\{M_t\left(1+i_t^M\right)+P_t b_t\left(1+i_{t+1}^K\right)-D_t\left(1+i_t^D+\bar{c}\right)\right\},$$

subject to

$$\begin{aligned} D_t &= M_t + P_t b_t, \\ D_t &\leq \ell(M_t + \rho P_t b_t), \end{aligned}$$

where $\rho < 1$ and $\ell \in (1, 1/\rho)$.

If the constraint is binding, then we must have

$$D_t = M_t + P_t b_t = \ell(M_t + \rho P_t b_t).$$

Hence,

$$\begin{aligned} P_t b_t &= \frac{\ell - 1}{1 - \ell\rho} M_t \\ D_t &= \frac{\ell\left(1-\rho\right)}{1-\ell\rho} M_t. \end{aligned}$$

Therefore, banks solve

$$M_t E_t \left\{ \left(1+i_t^M\right) + \frac{\ell-1}{1-\ell\rho}\left(1+i_{t+1}^K\right) - \frac{\ell(1-\rho)}{1-\ell\rho}\left(1+i_t^D+\bar{c}\right) \right\}.$$

In equilibrium, we must have

$\frac{1+i_t^S}{1+i_t^M}$

$$\frac{E_t i_{t+1}^K - i_t^M}{1+i_t^S} = \underbrace{\left(1+\frac{1-\ell\rho}{\ell-1}\right)}_{>1}\frac{i_t^D+\bar{c}-i_t^M}{1+i_t^S}$$

$$\frac{\left(E_t i_{t+1}^K - i_t^S\right) - \left(i_t^M - i_t^S\right)}{1+i_t^S} = \underbrace{\left(1+\frac{1-\ell\rho}{\ell-1}\right)}_{>1}\frac{\left(i_t^D - i_t^S\right)+\bar{c}-\left(i_t^M-i_t^S\right)}{1+i_t^S}.$$

This equation implies that $i_t^M < i_t^D + \bar{c}$ and $i_t^M < E_t i_{t+1}^K$. In the last one or two decades, the interest on deposits has been generally lower than the interest on reserves, i.e., $i_t^M - i_t^D > 0$, which implies $0 < i_t^M - i_t^D < \bar{c}$.

Compare this against our original formulation, (81).

B.2 Role of Financial Frictions

To see the role of financial frictions (FF), let's remove FFs and then rewrite the equations.

The bank problem without FFs/regulation can be written more simply as

$$\begin{aligned} & E_t\left\{M_t\left(1+i_t^M\right)+P_t b_t\left(1+i_{t+1}^K\right)-D_t\left(1+i_t^D\right)-A_t(1+i_t^S)\right\} \\ \text{st. } A_t &= M_t + P_t b_t - D_t. \end{aligned}$$

In equilibrium, $i_t^D < i_t^S$ because of deposits' liquidity premium, **so banks do not hold illiquid bonds**. The problem can be more simply written as

$$\max_{m,b} m_t\left(i_t^M - i_t^D\right) + b_t\left(E_t i_{t+1}^K - i_t^D\right).$$

FOCs:

$$i_t^M = E_t i_{t+1}^K = i_t^D < i_t^S, \text{ and } m_t > 0, d_t = m_t + b_t \tag{76}$$
$$i_t^M < E_t i_{t+1}^K = i_t^D < i_t^S, \text{ and } m_t = 0, d_t = b_t \tag{77}$$

A corollary is that the policy cannot set $i_t^M > i_t^S$ because then banks would have incentive to accumulate arbitrarily large quantity of reserves by issuing illiquid bonds. Similarly, if $i_t^M > i_t^D$, banks would have incentive to accumulate an arbitrarily large quantity of reserves by issuing deposits. Altogether, the reserves rate is a floor for the illiquid bond rate and deposit rate in this economy.

All equilibrium conditions of the simple model without FF, but reserves are used

We assume reserves are used, $m_t > 0$, which is a more realistic case. Also, it enables us to compare the MP framework that uses reserves with the MP framework that uses CBDC.

Steady State We can solve for the steady state similarly in 4 blocks as in the benchmark model, with only two differences: Equation (14) should be replaced by

$$i^M = i^D = i^K,$$

and (21) should be replaced by

$$d = m + b.$$

Log-linearized form Note that all equilibrium conditions are the same as in (98)-(113) except that Equations (102)–(104) should be replaced by the following:

$$\text{Bank FOC 1:} \quad i_t^M = i_t^D \tag{78}$$

$$\text{Bank FOC 2:} \quad i_t^M = E_t r_{t+1}^K + \left(1 + r^K\right) E_t\left(\pi_{t+1} - 1\right) \tag{79}$$

$$\text{Balance sheet identity:} \quad \widetilde{d} = \alpha_m \widetilde{m} + \left(1 - \alpha_m\right) \widehat{b}_t \tag{80}$$

where

$$\alpha_m \equiv \frac{M/P}{M/P + b}.$$

The third equation is the same as (104), but **notice that there is no ρ in the denominator of α_m.**

In principle, we can do the same exercises that we did in the benchmark economy where banks faced financial frictions.[15]

All equilibrium conditions of the simple model without FF and reserves

If $m_t = 0$, then $i_t^M < i_t^D = E_t i_{t+1}^K$, so i_t^M is irrelevant.

Steady State We can solve for the steady state similarly in 4 blocks as in the benchmark model, with only two differences:

Equation (14) should be replaced by

$$i^D = i^K,$$

and (21) should be replaced by

$$d_t = b_t.$$

[15] Note that here, i_t^M should not be too low because then $\widetilde{m}$ would be too low and $m > 0$ may be violated.

Banks' problem if banks cannot hold equity

$$\begin{aligned}\mathcal{R}_t &= E_t\left\{\bar{\Lambda}_{t+1}\Psi_{t+1}\right\}\\ &= E_t\left\{\bar{\Lambda}_{t+1}\left[\begin{array}{c} P_t b_t\left(1+i_{t+1}^K\right)+M_t\left(1+i_t^M\right)\\ -\left(1+i_t^D\right)D_t-\left(1+i_t^S\right)A_t\end{array}\right]\right\},\end{aligned}$$

where $\bar{\Lambda}_{t+1}$ is the nominal stochastic discount factor.

We write the Lagrangian for banks' problem as

$$\begin{aligned}&E_t\left\{\bar{\Lambda}_{t+1}\left[\begin{array}{c} M_t\left(1+i_t^M\right)+P_t b_t\left(1+i_{t+1}^K\right)\\ -\left(1+i_t^D\right)D_t-\left(1+i_t^S\right)\left(M_t+P_t b_t-D_t\right)\end{array}\right]\right\}\\ &+\lambda_t\left(-D_t+\ell M_t+\ell\rho P_t b_t\right),\end{aligned}$$

where λ_t the Lagrangian multiplier associated with the constraint.

FOCs are given by

$$\begin{aligned}M &: E_t\left\{\bar{\Lambda}_{t+1}\left(1+i_t^M\right)\right\}-E_t\left\{\bar{\Lambda}_{t+1}\left(1+i_t^S\right)\right\}+\lambda_t\ell=0\\ b &: E_t\left\{\bar{\Lambda}_{t+1}\left(1+i_{t+1}^K\right)\right\}-E_t\left\{\bar{\Lambda}_{t+1}\left(1+i_t^S\right)\right\}+\lambda_t\ell\rho=0\\ D &: E_t\left\{\bar{\Lambda}_{t+1}\left(1+i_t^D\right)\right\}-E_t\left\{\bar{\Lambda}_{t+1}\left(1+i_t^S\right)\right\}+\lambda_t=0\end{aligned}$$

$$E_t\left\{\bar{\Lambda}_{t+1}\left(i_t^S-i_t^D\right)\right\}=\frac{E_t\left\{\bar{\Lambda}_{t+1}\left(i_t^S-i_t^M\right)\right\}}{\ell}=\frac{E_t\left\{\bar{\Lambda}_{t+1}\left(i_t^S-i_{t+1}^K\right)\right\}}{\ell\rho}=\lambda_t$$

It is easily observed that these equations are identical to those with equity.

$$\frac{i_t^S-i_t^D}{1+i_t^S}=\frac{i_t^S-i_t^M}{\left(1+i_t^S\right)\ell}=\frac{i_t^S-E_t i_{t+1}^K}{\left(1+i_t^S\right)\rho\ell}=\lambda_t. \tag{81}$$

These equations relate the spread of deposits, reserves and the return on capital. A higher ℓ implies that the assets are better in backing liabilities, so the interest on reserves and the rate of return on capital both decrease.

Let's calculate the rents:

$$\begin{aligned}&E_t\left\{\begin{array}{c} P_t b_t\bar{\Lambda}_{t+1}\left(1+i_{t+1}^K\right)+M_t\bar{\Lambda}_{t+1}\left(1+i_t^M\right)\\ -\bar{\Lambda}_{t+1}\left(1+i_t^D\right)D_t-\bar{\Lambda}_{t+1}\left(1+i_t^S\right)A_t\end{array}\right\}\\ =\ &E_t\left\{\begin{array}{c} P_t b_t\left[E_t\left\{\bar{\Lambda}_{t+1}\left(1+i_t^S\right)\right\}-\lambda_t\ell\rho\right]+M_t\left[E_t\left\{\bar{\Lambda}_{t+1}\left(1+i_t^S\right)\right\}-\lambda_t\ell\right]\\ -\left[E_t\left\{\bar{\Lambda}_{t+1}\left(1+i_t^S\right)\right\}-\lambda_t\right]D_t-\bar{\Lambda}_{t+1}\left(1+i_t^S\right)\left(M_t+P_t b_t-D_t\right)\end{array}\right\}\\ =\ &E_t\left\{\begin{array}{c} E_t\left\{\bar{\Lambda}_{t+1}\left(1+i_t^S\right)\right\}P_t b_t-\lambda_t\ell\rho P_t b_t+E_t\left\{\bar{\Lambda}_{t+1}\left(1+i_t^S\right)\right\}M_t-\lambda_t\ell M_t\\ -E_t\left\{\bar{\Lambda}_{t+1}\left(1+i_t^S\right)\right\}D_t+\lambda_t D_t-\bar{\Lambda}_{t+1}\left(1+i_t^S\right)\left(M_t+P_t b_t-D_t\right)\end{array}\right\}\end{aligned}$$

$$=\lambda_t E_t\left\{-\ell\rho P_t b_t-\ell M_t+D_t\right\}=0$$

It is observed that the rents are zero in the case without equity.

C The Model with Cash

C.1 Derivation of HH's optimality conditions in the full model with cash

Modified utility function with cash is given by:[16]

$$
\begin{aligned}
& U\left(C_t, \frac{D_t}{P_t}, \frac{E_t}{P_t}, \frac{F_t}{P_t}, H_t\right) \\
= \; & \frac{1}{1-\frac{1}{\sigma}} \left(\begin{array}{c} C_t^{1-\frac{1}{\eta}} \\ +\omega_D \left((D_t/P_t)^{1-\frac{1}{\upsilon}} + \frac{\omega_{FD}}{\omega_D} (F_t/P_t)^{1-\frac{1}{\upsilon}} \right)^{\frac{1-\frac{1}{\eta}}{1-\frac{1}{\upsilon}}} \\ +\omega_E \left((E_t/P_t)^{1-\frac{1}{\varrho}} + \frac{\omega_{FE}}{\omega_E} (F_t/P_t)^{1-\frac{1}{\varrho}} \right)^{\frac{1-\frac{1}{\eta}}{1-\frac{1}{\varrho}}} \end{array} \right)^{\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\eta}}} - \frac{\psi}{1+\varphi} H_t^{1+\varphi}
\end{aligned}
$$

Optimality conditions:

$$
\begin{aligned}
C &: \frac{U_{C,t}}{P_t} = \lambda_t \\
J &: \frac{U_{J,t}}{P_t} = \lambda_t - \beta \boldsymbol{E} \lambda_{t+1} \left(1 + i_t^J\right) \text{ for } J \in \{D, E, F\} \\
S &: \lambda_t = \beta \boldsymbol{E}_t \lambda_{t+1} \left(1 + i_t^S\right) \\
\overset{J,S}{\Longrightarrow} \frac{U_{J,t}}{P_t} &= \lambda_t - \beta \boldsymbol{E} \lambda_{t+1} \left(1 + i_t^S - i_t^S + i_t^J\right) = \beta \boldsymbol{E} \lambda_{t+1} \left(i_t^S - i_t^J\right) = \lambda_t \frac{i_t^S - i_t^J}{1 + i_t^S} \\
H &: U_{H,t} = -\lambda_t W_t
\end{aligned}
$$

The FOCs can be summarized as

$$
\frac{U_C}{P_t} = \frac{U_D}{P_t} \frac{1+i^S}{i^S - i^D} = \frac{U_E}{P_t} \frac{1+i^S}{i^S} = \frac{U_F}{P_t} \frac{1+i^S}{i^S - i^F} = \frac{-U_H}{W_t} = \lambda_t = \beta \boldsymbol{E}_t \left[\lambda_{t+1}(1+i_t^S)\right]
$$

$$
1 = \frac{U_D}{U_C} \frac{1+i^S}{i^S - i^D} = \frac{U_E}{U_C} \frac{1+i^S}{i^S} = \frac{U_F}{U_C} \frac{1+i^S}{i^S - i^F} = \frac{1}{U_C} \frac{-U_H}{W_t/P_t} = \frac{\lambda_t P_t}{U_C}
$$

The money demand function for deposits and cash are given by

$$
\begin{aligned}
\frac{i^S - i^D}{1+i^S} &= \omega_D \left(\frac{P_t C_t}{D_t}\right)^{\frac{1}{\upsilon}} Q_{D,t}^{\frac{1}{\upsilon} - \frac{1}{\eta}} \\
\frac{i^S}{1+i^S} &= \omega_E \left(\frac{P_t C_t}{E_t}\right)^{\frac{1}{\varrho}} Q_{E,t}^{\frac{1}{\varrho} - \frac{1}{\eta}}
\end{aligned}
$$

[16] A cash-like CBDC is a perfect substitute for cash, which can be modeled by $\varrho = \infty$. A universal CBDC (as in Chiu and Davoodalhosseini (2023)) is a perfect substitute for cash and deposits: $\upsilon = \varrho = \infty$.

where $Q_{D,t} \equiv \left(\left(\frac{D_t}{P_tC_t}\right)^{1-\frac{1}{\upsilon}} + \frac{\omega_{FD}}{\omega_D}\left(\frac{F_t}{P_tC_t}\right)^{1-\frac{1}{\upsilon}}\right)^{\frac{1}{1-\frac{1}{\upsilon}}}$ and $Q_{E,t} \equiv \left(\left(\frac{E_t}{P_tC_t}\right)^{1-\frac{1}{\varrho}} + \frac{\omega_{FE}}{\omega_E}\left(\frac{F_t}{P_tC_t}\right)^{1-\frac{1}{\varrho}}\right)^{\frac{1}{1-\frac{1}{\varrho}}}$.

FOC for CBDC:

$$\begin{aligned}
1 &= \left(\frac{\omega_{FD}}{\omega_D}\left(\frac{D_t}{F_t}\right)^{\frac{1}{\upsilon}}\frac{U_D}{U_C} + \frac{\omega_{FE}}{\omega_E}\left(\frac{E_t}{F_t}\right)^{\frac{1}{\varrho}}\frac{U_E}{U_C}\right)\frac{1+i^S}{i^S-i^F} \\
\text{so } i^S - i^F &= \frac{\omega_{FD}}{\omega_D}\left(\frac{D_t}{F_t}\right)^{\frac{1}{\upsilon}}\left(i^S - i^D\right) + \frac{\omega_{FE}}{\omega_E}\left(\frac{E_t}{F_t}\right)^{\frac{1}{\varrho}} i^S \\
\text{or } \frac{i^S - i^F}{1+i^S} &= \omega_{FD}\left(\frac{P_tC_t}{F_t}\right)^{\frac{1}{\upsilon}} Q_{D,t}^{\frac{1}{\upsilon}-\frac{1}{\eta}} + \omega_{FE}\left(\frac{P_tC_t}{F_t}\right)^{\frac{1}{\varrho}} Q_{E,t}^{\frac{1}{\varrho}-\frac{1}{\eta}} \qquad (82)
\end{aligned}$$

For labor:

$$\begin{aligned}
\psi H_t^{\varphi} &= \lambda_t W_t = \frac{W_t}{P_t} C_t^{-\frac{1}{\eta}} XX^{\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\eta}}-1} = \frac{W_t}{P_t} C_t^{-\frac{1}{\eta}}\left(\frac{Q_t}{C_t^{\frac{1}{\eta}}}\right)^{(1-\eta)\left(\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\eta}}-1\right)} = \frac{W_t}{P_t} C_t^{-\frac{1}{\sigma}} Q_t^{\frac{\eta}{\sigma}-1} \\
&\Rightarrow\ Q_t^{1-\frac{\eta}{\sigma}} C_t^{\frac{1}{\sigma}} \psi H_t^{\varphi} = \frac{W_t}{P_t}
\end{aligned}$$

Note that $Q_t \equiv \left(\frac{XX_t}{C_t^{\frac{\eta-1}{\eta}}}\right)^{\frac{1}{1-\eta}} \Rightarrow XX_t = \left(\frac{Q_t}{C_t^{\frac{1}{\eta}}}\right)^{1-\eta}$.

Demand for assets:

Demand for bonds:

$$\begin{aligned}
\lambda_t &= C_t^{-\frac{1}{\eta}} P_t^{-1} XX_t^{\frac{-\frac{1}{\sigma}+\frac{1}{\eta}}{1-\frac{1}{\eta}}}, \lambda_{t+1} = C_{t+1}^{-\frac{1}{\eta}} P_{t+1}^{-1} XX_{t+1}^{\frac{-\frac{1}{\sigma}+\frac{1}{\eta}}{1-\frac{1}{\eta}}}, \lambda_t = \beta \boldsymbol{E}_t\left[\lambda_{t+1}(1+i_t^S)\right] \\
&\Rightarrow\ 1 = \beta \boldsymbol{E}_t\left[\frac{\lambda_{t+1}}{\lambda_t}(1+i_t^S)\right] = \beta \boldsymbol{E}_t\left[\left(\left(\frac{XX_{t+1}}{XX_t}\right)^{\frac{1}{1-\eta}}\right)^{\frac{\eta}{\sigma}-1}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\eta}}\frac{P_t}{P_{t+1}}\right](1+i_t^S) \\
&= \beta \boldsymbol{E}_t\left[\frac{C_{t+1}^{\frac{1}{\eta}-\frac{1}{\sigma}} Q_{t+1}^{\frac{\eta}{\sigma}-1}}{C_t^{\frac{1}{\eta}-\frac{1}{\sigma}} Q_t^{\frac{\eta}{\sigma}-1}}\frac{C_{t+1}^{-\frac{1}{\eta}}}{C_t^{-\frac{1}{\eta}}}\frac{P_t}{P_{t+1}}\right](1+i_t^S) \\
&\Rightarrow\ \text{Euler Eq: } \beta E_t\left[\left(\frac{Q_{t+1}}{Q_t}\right)^{\frac{\eta}{\sigma}-1}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\sigma}}\frac{P_t}{P_{t+1}}\right]\left(1+i_t^S\right) = 1 \qquad (83)
\end{aligned}$$

We have used a change of variables from XX to Q:

$$Q_t \equiv \left(\frac{XX_t}{C_t^{\frac{\eta-1}{\eta}}}\right)^{\frac{1}{1-\eta}} = \left(\begin{array}{c} 1+\omega_D\left(\frac{(D_t/P_t)^{1-\frac{1}{\upsilon}}+\frac{\omega_{FD}}{\omega_D}(F_t/P_t)^{1-\frac{1}{\upsilon}}}{C_t^{1-\frac{1}{\upsilon}}}\right)^{\frac{1-\frac{1}{\eta}}{1-\frac{1}{\upsilon}}} \\ +\omega_E\left(\frac{(E_t/P_t)^{1-\frac{1}{\varrho}}+\frac{\omega_{FE}}{\omega_E}(F_t/P_t)^{1-\frac{1}{\varrho}}}{C_t^{1-\frac{1}{\varrho}}}\right)^{\frac{1-\frac{1}{\eta}}{1-\frac{1}{\varrho}}} \end{array}\right)^{\frac{1}{1-\eta}}$$

$$Q_t = \left(\begin{array}{c} 1+\omega_D\left(\frac{(D_t/P_t)^{1-\frac{1}{\upsilon}}+\frac{\omega_{FD}}{\omega_D}(F_t/P_t)^{1-\frac{1}{\upsilon}}}{C_t^{1-\frac{1}{\upsilon}}}\right)^{\frac{1-\frac{1}{\eta}}{1-\frac{1}{\upsilon}}} \\ +\omega_E\left(\frac{(E_t/P_t)^{1-\frac{1}{\varrho}}+\frac{\omega_{FE}}{\omega_E}(F_t/P_t)^{1-\frac{1}{\varrho}}}{C_t^{1-\frac{1}{\varrho}}}\right)^{\frac{1-\frac{1}{\eta}}{1-\frac{1}{\varrho}}} \end{array}\right)^{\frac{1}{1-\eta}}$$

$$Q_t = \left(1+\omega_D Q_{D,t}^{1-\frac{1}{\eta}}+\omega_E Q_{E,t}^{1-\frac{1}{\eta}}\right)^{\frac{1}{1-\eta}}.$$

Now, we derive similar equations for deposits, cash and CBDC:

$$1 = \beta E_t\left[\left(\frac{Q_{t+1}}{Q_t}\right)^{\frac{\eta}{\sigma}-1}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\sigma}}\frac{P_t}{P_{t+1}}\right]\left(1+i_t^D+\frac{i_t^S-i_t^D}{1+i_t^S}\left(1+i_t^S\right)\right)$$

$$1 = \beta E_t\left[\left(\frac{Q_{t+1}}{Q_t}\right)^{\frac{\eta}{\sigma}-1}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\sigma}}\frac{P_t}{P_{t+1}}\right]\left(1+i_t^D\right)+\frac{i_t^S-i_t^D}{1+i_t^S}$$

$$1 = \beta E_t\left[\left(\frac{Q_{t+1}}{Q_t}\right)^{\frac{\eta}{\sigma}-1}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\sigma}}\frac{P_t}{P_{t+1}}\right]\left(1+i_t^D\right)$$
$$+\left(\frac{P_tC_t}{D_t}\right)^{\frac{1}{\upsilon}}\omega_D\left[\left(\frac{D_t}{P_tC_t}\right)^{1-\frac{1}{\upsilon}}+\frac{\omega_{FD}}{\omega_D}\left(\frac{F_t}{P_tC_t}\right)^{1-\frac{1}{\upsilon}}\right]^{\frac{\frac{1}{\upsilon}-\frac{1}{\eta}}{1-\frac{1}{\upsilon}}}$$

Therefore:

Deposit demand: $$1 = \beta E_t\left[\left(\frac{Q_{t+1}}{Q_t}\right)^{\frac{\eta}{\sigma}-1}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\sigma}}\frac{P_t}{P_{t+1}}\right]\left(1+i_t^D\right)+\omega_D\left(\frac{P_tC_t}{D_t}\right)^{\frac{1}{\upsilon}}Q_{D,t}^{\frac{1}{\upsilon}-\frac{1}{\eta}},$$

Cash demand: $$1 = \beta E_t\left[\left(\frac{Q_{t+1}}{Q_t}\right)^{\frac{\eta}{\sigma}-1}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\sigma}}\frac{P_t}{P_{t+1}}\right]\left(1+i_t^E\right)+\omega_E\left(\frac{P_tC_t}{E_t}\right)^{\frac{1}{\varrho}}Q_{E,t}^{\frac{1}{\varrho}-\frac{1}{\eta}},$$

CBDC demand: $$1 = \beta E_t\left[\left(\frac{Q_{t+1}}{Q_t}\right)^{\frac{\eta}{\sigma}-1}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\sigma}}\frac{P_t}{P_{t+1}}\right]\left(1+i_t^F\right)$$
$$+\omega_{FD}\left(\frac{P_tC_t}{F_t}\right)^{\frac{1}{\upsilon}}Q_{D,t}^{\frac{1}{\upsilon}-\frac{1}{\eta}}+\omega_{FE}\left(\frac{P_tC_t}{F_t}\right)^{\frac{1}{\varrho}}Q_{E,t}^{\frac{1}{\varrho}-\frac{1}{\eta}}.$$

All equilibrium conditions of the full model with cash

$$\text{Illiquid bond demand:} \quad \beta E_t \left[\left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{\eta}{\sigma}-1} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\sigma}} \frac{P_t}{P_{t+1}} \right] \left(1 + i_t^S\right) = 1$$

$$\text{Deposit demand:} \quad \frac{i_t^S - i_t^D}{1 + i_t^S} = \omega_D V_{D,t}^{\frac{1}{\upsilon}} V_{FD,t}^{-\frac{1}{\upsilon}+\frac{1}{\eta}}$$

$$\text{Cash demand:} \quad \frac{i_t^S - i_t^E}{1 + i_t^S} = \omega_E V_{E,t}^{\frac{1}{\varrho}} V_{FE,t}^{-\frac{1}{\varrho}+\frac{1}{\eta}}$$

$$\text{CBDC demand:} \quad \frac{i_t^S - i_t^F}{1 + i_t^S} = \omega_{FD} V_{F,t}^{\frac{1}{\upsilon}} V_{FD,t}^{-\frac{1}{\upsilon}+\frac{1}{\eta}} + \omega_{FE} V_{F,t}^{\frac{1}{\varrho}} V_{FE,t}^{-\frac{1}{\varrho}+\frac{1}{\eta}}$$

$$\text{Production: } Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$

$$\text{Market clearing: } Y_t = C_t + I_t + \frac{\kappa}{2} \left(\frac{P_t}{P_{t-1}} - 1 \right)^2$$

$$K_{t+1} = b_t.$$

$$K_{t+1} = I_t + (1-\delta) K_t.$$

$$\text{Labor demand:} \quad w_t = p_{mt}(1-\alpha) \frac{Y_t}{L_t}$$

$$\text{Capital demand:} \quad 1 + r_t^K = \alpha p_{mt} \frac{Y_t}{K_t} + 1 - \delta$$

$$\text{Labor supply:} \quad w_t = Q_t^{1-\frac{\eta}{\sigma}} C_t^{\frac{1}{\sigma}} \psi L_t^{\varphi}$$

$$\rightarrow p_{mt} Y_t = \frac{\left(\delta + r_t^K\right) K_t}{\alpha} = Q_t^{1-\frac{\eta}{\sigma}} C_t^{\frac{1}{\sigma}} \psi \frac{L_t^{1+\varphi}}{(1-\alpha)}$$

$$\text{Optimal pricing:} \quad \left[\frac{\epsilon - 1}{\epsilon} - p_{mt} \right] \frac{\epsilon Y_t}{\kappa} + \frac{P_t}{P_{t-1}} \left(\frac{P_t}{P_{t-1}} - 1 \right) = E_t \left[\Lambda_{t,t+1} \frac{P_{t+1}}{P_t} \left(\frac{P_{t+1}}{P_t} - 1 \right) \right]$$

$$\text{Binding leverage constraint:} \quad D_t = \ell (M_t + \rho P_t b_t)$$

$$\text{Bank FOCs:} \quad \frac{i_t^S - i_t^D}{1 + i_t^S} = \frac{i_t^S - i_t^M}{(1 + i_t^S)\, \ell} = \frac{i_t^S - E_t r_{t+1}^K - \left(1 + r^K\right) E_t \left(\pi_{t+1} - 1\right)}{(1 + i_t^S)\, \rho \ell}$$

where

$$\text{Velocity:} V_{J,t} \equiv \frac{P_t C_t}{J_t} \text{ for } J \in \{D, E, F\}$$

$$\begin{aligned}
V_{FD,t} &\equiv \left(V_{D,t}^{-(1-\frac{1}{\upsilon})} + \frac{\omega_{FD}}{\omega_D} V_{F,t}^{-\left(1-\frac{1}{\upsilon}\right)} \right)^{\frac{-1}{1-\frac{1}{\upsilon}}} \\
V_{FE,t} &\equiv \left(V_{E,t}^{-\left(1-\frac{1}{\varrho}\right)} + \frac{\omega_{FE}}{\omega_E} V_{F,t}^{-\left(1-\frac{1}{\varrho}\right)} \right)^{\frac{-1}{1-\frac{1}{\varrho}}} \\
Q_t &\equiv \left(1 + \omega_D V_{FD,t}^{\frac{1}{\eta}-1} + \omega_E V_{FE,t}^{\frac{1}{\eta}-1} \right)^{\frac{1}{1-\eta}}
\end{aligned}$$

Steady state equations with cash

Unknowns:

- Output, consumption and labor: Y, C, L
- Deposits, cash, CBDC and reserves balances: D, E, F, M
- Real assets: b
- Rates: i^K, i^D

Note that the nominal and real interest rates are equal because the inflation rate is zero, i.e., $i^K = r^K$ and $i^D = r^D$. We now derive the steady state values:

$$\begin{aligned}
\text{Intermediate good price:} &\qquad p_m = \frac{\epsilon - 1}{\epsilon}\ , \\
\text{Illiquid bond demand:} &\qquad \beta\left(1 + i^S\right) = 1.
\end{aligned}$$

The price of the intermediate good is $\frac{\epsilon-1}{\epsilon}$ in terms of the final good. This gives the markup of $\frac{1}{\epsilon-1}$, which is simply due to the market power. We solve the model in four blocks below. This solution method also reveals the transmission of monetary policy in the steady state.

Block 1: Given i^M, we can solve for i^K and i^D:

$$\frac{i^S - i^D}{1 + i^S} = \frac{i^S - i^M}{(1 + i^S)\,\ell} = \frac{i^S - i^K}{(1 + i^S)\,\rho\ell}. \tag{84}$$

Block 2: Given i^D from block 1 and $i^E = 0$ and i^F from policy, we can pin down V_D, V_E

and V_F:

$$\text{Deposit demand:} \quad \frac{i^S - i^D}{1+i^S} = \omega_D V_D^{\frac{1}{\upsilon}} Q_D^{-\frac{1}{\upsilon}+\frac{1}{\eta}}, \tag{85}$$

$$\text{Cash demand:} \quad \frac{i^S - i^E}{1+i^S} = \omega_E V_E^{\frac{1}{\varrho}} Q_E^{-\frac{1}{\varrho}+\frac{1}{\eta}}, \tag{86}$$

$$\text{CBDC demand:} \quad \frac{i^S - i^F}{1+i^S} = \omega_{FD} V_F^{\frac{1}{\upsilon}} Q_D^{-\frac{1}{\upsilon}+\frac{1}{\eta}} + \omega_{FE} V_F^{\frac{1}{\varrho}} Q_E^{-\frac{1}{\varrho}+\frac{1}{\eta}}. \tag{87}$$

Note that Q_D and Q_E are functions of V_J's. We can then calculate Q.

Moreover, we have

$$b = K = \frac{I}{\delta}.$$

Block 3: Given i^K (from block 1) and Q (from block 2), the following four equations pin down Y, L, C and b:

$$\begin{aligned}
Y &= C + \delta K, \\
p_m &= \frac{\epsilon-1}{\epsilon} = \frac{\left(i^K+\delta\right)K}{\alpha Y} \rightarrow K = \frac{\epsilon-1}{\epsilon}\frac{\alpha Y}{i^K+\delta}, \\
Y &= A K^{\alpha} L^{1-\alpha}, \\
\frac{\epsilon-1}{\epsilon} Y &= Q^{1-\frac{\eta}{\sigma}} C^{\frac{1}{\sigma}} \psi \frac{L^{1+\varphi}}{1-\alpha}.
\end{aligned} \tag{88}$$

We can now calculate Y as a function of C and K and then use the market clearing condition:

$$Y^{\frac{1+\varphi}{1-\alpha}-1} = \frac{\epsilon-1}{\epsilon}\frac{(1-\alpha)A^{\frac{1+\varphi}{1-\alpha}} K^{\alpha\frac{1+\varphi}{1-\alpha}}}{Q^{1-\frac{\eta}{\sigma}} C^{\frac{1}{\sigma}} \psi},$$

$$Y = C + \delta K \rightarrow C = Y\left(1 - \frac{\epsilon-1}{\epsilon}\frac{\alpha\delta}{i^K+\delta}\right). \tag{89}$$

Given that K and C are now given in terms of Y from (88) and (89), we obtain

$$Y^{\varphi+\frac{1}{\sigma}} = \frac{\alpha^{\frac{\alpha(1+\varphi)}{1-\alpha}}(1-\alpha)A^{\frac{1+\varphi}{1-\alpha}}\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}-1}}{\psi Q^{1-\frac{\eta}{\sigma}}\left(1-\frac{\epsilon-1}{\epsilon}\frac{\alpha\delta}{i^K+\delta}\right)^{\frac{1}{\sigma}}\left(i^K+\delta\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}}}. \tag{90}$$

Log-linearization around the zero-inflation-rate steady state in the full model with cash

We generally use small-case letters for log-linearized form, i.e., $\widehat{c}_t$ is log-lin of C_t.

Definition of constants:

$$
\begin{aligned}
\widetilde{x} &= \widehat{x}_t - \widehat{p}_t \text{ for } x \in \{d, e, f, m\} \\
\alpha_{FD} &\equiv \frac{\omega_{FD} V_F^{\frac{1}{\upsilon}} V_{FD}^{\frac{1}{\eta}-\frac{1}{\upsilon}}}{\omega_{FD} V_F^{\frac{1}{\upsilon}} V_{FD}^{\frac{1}{\eta}-\frac{1}{\upsilon}} + \omega_{FE} V_E^{\frac{1}{\varrho}} V_{FE}^{\frac{1}{\eta}-\frac{1}{\varrho}}} \\
\beta_J &\equiv \frac{V_D^{-(1-\frac{1}{\upsilon})}}{V_D^{-(1-\frac{1}{\upsilon})} + \frac{\omega_{FD}}{\omega_D} V_F^{-\left(1-\frac{1}{\upsilon}\right)}}, \\
\beta_E &\equiv \frac{V_E^{-(1-\frac{1}{\varrho})}}{V_E^{-(1-\frac{1}{\varrho})} + \frac{\omega_{FE}}{\omega_E} V_F^{-\left(1-\frac{1}{\varrho}\right)}}.
\end{aligned}
$$

$$
\alpha_m \equiv \frac{M/P}{M/P + \rho b}.
$$

$$
\alpha_c \equiv \frac{C}{Y}, \text{ and } \alpha_y \equiv \frac{\alpha \frac{\epsilon-1}{\epsilon} \frac{Y}{K}}{\alpha \frac{\epsilon-1}{\epsilon} \frac{Y}{K} + 1 - \delta}.
$$

$$
\alpha_{JJ} \equiv \frac{\omega_D V_{FD}^{\frac{1}{\eta}-1}}{1 + \omega_D V_{FD}^{\frac{1}{\eta}-1} + \omega_E V_{FE}^{\frac{1}{\eta}-1}}, J \in \{D, E\}
$$

Derivations of log-linearized HHs' FOCs:

We start with HHs' FOC:

$$
\text{FOC:} \frac{i_t^S - i_t^F}{1 + i_t^S} = \omega_{FD} V_{F,t}^{\frac{1}{\upsilon}} V_{FD,t}^{\frac{1}{\eta}-\frac{1}{\upsilon}} + \omega_{FE} V_{E,t}^{\frac{1}{\varrho}} V_{FE,t}^{\frac{1}{\eta}-\frac{1}{\varrho}}
$$

The LHS:

$$
\begin{aligned}
\log\left(\frac{i_t^S - i_t^F}{1 + i_t^S} / \frac{i^S - i^F}{1 + i^S}\right) &= \log\left(\frac{i_t^S - i_t^F}{i^S - i^F}\right) - \log\left(\frac{1 + i_t^S}{1 + i^S}\right) \\
\log\left(\frac{i_t^S - i_t^F}{i^S - i^F}\right) &= \log\left(1 + \frac{i_t^S - i^S - \left(i_t^F - i^F\right)}{i^S - i^F}\right) \approx \frac{i_t^S - i_t^F}{i^S - i^F} - 1 \\
\log\left(\frac{1 + i_t^S}{1 + i^S}\right) &= \log\left(1 + \frac{i_t^S - i^S}{1 + i^S}\right) \approx \frac{1}{i^S - i^F} \frac{\left(i_t^S - i^S\right)\left(i^S - i^F\right)}{\left(1 + i^S\right)} \approx 0
\end{aligned}
$$

Note that the last approximation is due to the fact that $\left(i_t^S - i^S\right)\left(i^S - i^F\right)$ is a second-order term while $i_t^S - i_t^F$ is a first-order one, so we can ignore the former against the latter. Therefore,

$$
\log\left(\frac{i_t^S - i_t^F}{1 + i_t^S} / \frac{i^S - i^F}{1 + i^S}\right) \approx \frac{i_t^S - i_t^F}{i^S - i^F} - 1.
$$

The RHS:

$$
\begin{aligned}
& \log\left(\frac{\omega_{FD}V_{F,t}^{\frac{1}{\upsilon}}V_{FD,t}^{\frac{1}{\eta}-\frac{1}{\upsilon}}+\omega_{FE}V_{E,t}^{\frac{1}{\varrho}}V_{FE,t}^{\frac{1}{\eta}-\frac{1}{\varrho}}}{\omega_{FD}V_{F}^{\frac{1}{\upsilon}}V_{FD}^{\frac{1}{\eta}-\frac{1}{\upsilon}}+\omega_{FE}V_{E}^{\frac{1}{\varrho}}V_{FE}^{\frac{1}{\eta}-\frac{1}{\varrho}}}\right) \\
\approx\ & \alpha_{FD}\left(\frac{1}{\upsilon}\widehat{V_{F,t}}+\left(\frac{1}{\eta}-\frac{1}{\upsilon}\right)\widehat{V_{FD,t}}\right)+(1-\alpha_{FD})\left(\frac{1}{\varrho}\widehat{V_{E,t}}+\left(\frac{1}{\eta}-\frac{1}{\varrho}\right)\widehat{V_{FE,t}}\right) \\
=\ & \alpha_{FD}\left(\frac{1}{\upsilon}\left(\widehat{p}_t+\widehat{c}_t-\widehat{f}_t\right)+\left(\frac{1}{\eta}-\frac{1}{\upsilon}\right)\left(\beta_D\left(\widehat{p}_t+\widehat{c}_t-\widehat{d}_t\right)+(1-\beta_D)\left(\widehat{p}_t+\widehat{c}_t-\widehat{f}_t\right)\right)\right) \\
& +(1-\alpha_{FD})\left(\frac{1}{\varrho}\left(\widehat{p}_t+\widehat{c}_t-\widehat{f}_t\right)+\left(\frac{1}{\eta}-\frac{1}{\varrho}\right)\left(\beta_E\left(\widehat{p}_t+\widehat{c}_t-\widehat{e}_t\right)+(1-\beta_E)\left(\widehat{p}_t+\widehat{c}_t-\widehat{f}_t\right)\right)\right) \\
=\ & \left[\begin{array}{c}\alpha_{FD}\left(\frac{1}{\upsilon}+\left(\frac{1}{\eta}-\frac{1}{\upsilon}\right)(1-\beta_D)\right) \\ +(1-\alpha_{FD})\left(\frac{1}{\varrho}+\left(\frac{1}{\eta}-\frac{1}{\varrho}\right)(1-\beta_E)\right)\end{array}\right]\left(\widehat{p}_t+\widehat{c}_t-\widehat{f}_t\right) \\
& +\alpha_{FD}\left(\frac{1}{\eta}-\frac{1}{\upsilon}\right)\beta_D\left(\widehat{p}_t+\widehat{c}_t-\widehat{d}_t\right)+(1-\alpha_{FD})\left(\frac{1}{\eta}-\frac{1}{\varrho}\right)\beta_E\left(\widehat{p}_t+\widehat{c}_t-\widehat{e}_t\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{i_t^S-i_t^F}{i^S-i^F}-1 &= \frac{1}{\eta}\left(\widehat{p}_t+\widehat{c}_t\right)-\alpha_{FD}\left(\frac{1}{\eta}-\frac{1}{\upsilon}\right)\beta_D\widehat{d}_t-(1-\alpha_{FD})\left(\frac{1}{\eta}-\frac{1}{\varrho}\right)\beta_E\widehat{e}_t \\
& \quad -\left[\alpha_{FD}\left(\frac{1-\beta_D}{\eta}+\frac{\beta_D}{\upsilon}\right)+(1-\alpha_{FD})\left(\frac{1-\beta_E}{\eta}+\frac{\beta_E}{\varrho}\right)\right]\widehat{f}_t \qquad (91)
\end{aligned}
$$

where we have used

$$
\begin{aligned}
\alpha_{FD} &\equiv \frac{\omega_{FD}V_F^{\frac{1}{\upsilon}}V_{FD}^{\frac{1}{\eta}-\frac{1}{\upsilon}}}{\omega_{FD}V_F^{\frac{1}{\upsilon}}V_{FD}^{\frac{1}{\eta}-\frac{1}{\upsilon}}+\omega_{FE}V_E^{\frac{1}{\varrho}}V_{FE}^{\frac{1}{\eta}-\frac{1}{\varrho}}} \\
\beta_D &\equiv \frac{V_D^{-(1-\frac{1}{\upsilon})}}{V_D^{-(1-\frac{1}{\upsilon})}+\frac{\omega_{FD}}{\omega_D}V_F^{-\left(1-\frac{1}{\upsilon}\right)}} \\
\beta_E &\equiv \frac{V_E^{-(1-\frac{1}{\varrho})}}{V_E^{-(1-\frac{1}{\varrho})}+\frac{\omega_{FE}}{\omega_E}V_F^{-\left(1-\frac{1}{\varrho}\right)}} \\
\widehat{V}_{J,t} &= \log\left(\frac{V_{J,t}}{V_J}\right)=\widehat{p}_t+\widehat{c}_t-\widehat{j}_t \text{ for } J\in\{D,E\} \\
\widehat{V}_{FJ,t} &= \beta_J\widehat{V}_{J,t}+(1-\beta_J)\widehat{V}_{F,t}=\beta_J\left(\widehat{p}_t+\widehat{c}_t-\widehat{j}_t\right)+(1-\beta_J)\left(\widehat{p}_t+\widehat{c}_t-\widehat{f}_t\right) \text{ for } J\in\{D,E\}
\end{aligned}
$$

Similarly, we can derive "money demand" for deposits and cash:

$$
\begin{aligned}
\frac{i_t^S - i_t^D}{i^S - i^D} - 1 &= \frac{1}{\upsilon}\widehat{V}_{D,t} + \left(\frac{1}{\eta} - \frac{1}{\upsilon}\right)\widehat{V}_{FD,t} \\
&= \frac{1}{\upsilon}\left[\widehat{p}_t + \widehat{c}_t - \widehat{d}_t\right] + \left(\frac{1}{\eta} - \frac{1}{\upsilon}\right)\left[\beta_D\left(\widehat{p}_t + \widehat{c}_t - \widehat{d}_t\right) + (1-\beta_D)\left(\widehat{p}_t + \widehat{c}_t - \widehat{f}_t\right)\right] \rightarrow \\
\frac{i_t^S - i_t^D}{i^S - i^D} - 1 &= \frac{1}{\eta}\left(\widehat{p}_t + \widehat{c}_t\right) - \left(\frac{1-\beta_D}{\upsilon} + \frac{\beta_D}{\eta}\right)\widehat{d}_t - \left(\frac{1}{\eta} - \frac{1}{\upsilon}\right)(1-\beta_D)\widehat{f}_t \qquad (92)
\end{aligned}
$$

$$
\begin{aligned}
\frac{i_t^S}{i^S} - 1 &= \frac{1}{\varrho}\widehat{V}_{E,t} + \left(\frac{1}{\eta} - \frac{1}{\varrho}\right)\widehat{V}_{FE,t} \\
&= \frac{1}{\varrho}\left[\widehat{p}_t + \widehat{c}_t - \widehat{e}_t\right] + \left(\frac{1}{\eta} - \frac{1}{\varrho}\right)\left[\beta_E\left(\widehat{p}_t + \widehat{c}_t - \widehat{e}_t\right) + (1-\beta_E)\left(\widehat{p}_t + \widehat{c}_t - \widehat{f}_t\right)\right] \rightarrow \\
\frac{i_t^S}{i^S} - 1 &= \frac{1}{\eta}\left(\widehat{p}_t + \widehat{c}_t\right) - \left(\frac{1-\beta_E}{\varrho} + \frac{\beta_E}{\eta}\right)\widehat{e}_t - \left(\frac{1}{\eta} - \frac{1}{\varrho}\right)(1-\beta_E)\widehat{f}_t \qquad (93)
\end{aligned}
$$

$$
\begin{aligned}
Q_t &\equiv \left(1 + \omega_D V_{FD,t}^{\frac{1}{\eta}-1} + \omega_E V_{FE,t}^{\frac{1}{\eta}-1}\right)^{\frac{1}{1-\eta}} \\
\widehat{q}_t &= \frac{1}{\eta}\left(\alpha_{DD}\widehat{V}_{FD,t} + \alpha_{EE}\widehat{V}_{FE,t}\right) \\
&= \frac{1}{\eta}\left(\alpha_{DD}\left(\beta_D\widehat{V}_{D,t} + (1-\beta_D)\widehat{V}_{F,t}\right) + \alpha_{EE}\left(\beta_E\widehat{V}_{E,t} + (1-\beta_E)\widehat{V}_{F,t}\right)\right) \\
&= \frac{1}{\eta}\begin{pmatrix} \alpha_{DD}\left(\beta_D\left(\widehat{p}_t + \widehat{c}_t - \widehat{d}_t\right) + (1-\beta_D)\left(\widehat{p}_t + \widehat{c}_t - \widehat{f}_t\right)\right) \\ +\alpha_{EE}\left(\beta_E\left(\widehat{p}_t + \widehat{c}_t - \widehat{e}_t\right) + (1-\beta_E)\left(\widehat{p}_t + \widehat{c}_t - \widehat{f}_t\right)\right) \end{pmatrix} \\
&= \frac{1}{\eta}\left(\left(\alpha_{DD} + \alpha_{EE}\right)\left(\widehat{p}_t + \widehat{c}_t\right) - \begin{pmatrix} +\alpha_{DD}\beta_D\widehat{d}_t + \alpha_{EE}\beta_E\widehat{e}_t \\ +\left[\alpha_{DD}(1-\beta_D) + \alpha_{EE}(1-\beta_E)\right]\widehat{f}_t \end{pmatrix}\right) \text{(QLL)}
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_{DD} &\equiv \frac{\omega_D V_{FD}^{\frac{1}{\eta}-1}}{1 + \omega_D V_{FD}^{\frac{1}{\eta}-1} + \omega_E V_{FE}^{\frac{1}{\eta}-1}}, \\
\alpha_{EE} &\equiv \frac{\omega_E V_{FE}^{\frac{1}{\eta}-1}}{1 + \omega_D V_{FD}^{\frac{1}{\eta}-1} + \omega_E V_{FE}^{\frac{1}{\eta}-1}}.
\end{aligned}
$$

Now we log-linearize the Euler equation for illiquid bond demand:

$$\begin{aligned}
E_t\left[\left(\frac{Q_{t+1}}{Q_t}\right)^{\frac{\eta}{\sigma}-1}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\sigma}}\frac{P_t}{P_{t+1}}\right]\left(1+\frac{i_t^S-i^S}{1+i^S}\right) &= 1\\
\left(\frac{\eta}{\sigma}-1\right)\left(E_t\left[\widehat{q}_{t+1}\right]-\widehat{q}_t\right)-\frac{1}{\sigma}\left(E_t\left[\widehat{c}_{t+1}\right]-\widehat{c}_t\right)-E_t\left[\widehat{p}_{t+1}\right]+\frac{i_t^S-i^S}{1+i^S} &= 0\\
\left(\frac{\eta}{\sigma}-1\right)\left(E_t\left[\widehat{q}_{t+1}\right]-\widehat{q}_t\right)-\frac{1}{\sigma}\left(E_t\left[\widehat{c}_{t+1}\right]-\widehat{c}_t\right)-E_t\left[\Delta\widehat{p}_{t+1}\right]+\beta i_t^S+\beta-1 &= 0
\end{aligned}$$

$$\widehat{c}_t = E_t\left[\widehat{c}_{t+1}\right]-\sigma\left(\beta i_t^S-E_t\left[\Delta\widehat{p}_{t+1}\right]+\beta-1\right)+(\sigma-\eta)\left(E_t\left[\widehat{q}_{t+1}\right]-\widehat{q}_t\right)$$

Log-lin of Philips curve:

Using $x_t \approx x(1+\widehat{x}_t)$, we can write $\Lambda_{t,t+1}\approx\beta(1+\widehat{\Lambda}_{t+1})$ and $\Lambda_{t,t+1}\equiv\beta\frac{U_{c,t+1}}{U_{c,t}}=\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\eta}}$, so $\widehat{\Lambda}_{t+1}=-\frac{1}{\eta}\left(\widehat{c}_{t+1}-\widehat{c}_t\right)$.

$$E_t\left[\Lambda_{t,t+1}\pi_{t+1}\right]=E_t\left[\Lambda_{t,t+1}\pi_{t+1}\right]\approx E_t\left[\beta(1-\frac{1}{\eta}\left(\widehat{c}_{t+1}-\widehat{c}_t\right))\pi_{t+1}\right]\approx E_t\left[\beta\pi_{t+1}\right]$$

We have

$$\begin{aligned}
\left[\frac{\epsilon-1}{\epsilon}-p_{mt}\right]\frac{\epsilon Y_t}{\kappa}+\frac{P_t}{P_{t-1}}\left(\frac{P_t}{P_{t-1}}-1\right) &= E_t\left[\Lambda_{t,t+1}\frac{P_{t+1}}{P_t}\left(\frac{P_{t+1}}{P_t}-1\right)\right]\\
\left[\frac{\epsilon-1}{\epsilon}-p_m(1+\widehat{p}_{mt})\right]\epsilon Y\left(1+\widehat{y}_t\right)+\kappa\left(1+\widehat{\pi}_t\right)\widehat{\pi}_t &\approx \kappa E_t\left[\Lambda_{t,t+1}\left(1+\widehat{\pi}_{t+1}\right)\widehat{\pi}_{t+1}\right]\\
\widehat{\pi}_t &\approx \frac{(\epsilon-1)Y}{\kappa}\widehat{p}_{mt}+\beta E_t\left[\widehat{\pi}_{t+1}\right] \qquad (94)
\end{aligned}$$

where we used the following notation: $\widehat{\pi}_t\equiv\Delta\hat{p}_t\equiv\log\frac{P_t}{P_{t-1}}$.

Rest of equations:

$$Y_t = A_tK_t^{\alpha}L_t^{1-\alpha}\rightarrow\widehat{y}_t=\alpha\widehat{k}_t+(1-\alpha)\,\widehat{l}_t$$

$$Y_t = C_t+I_t+\frac{\kappa}{2}\left(\frac{P_t}{P_{t-1}}-1\right)^2\rightarrow\widehat{y}_t=\alpha_c\widehat{c}_t+(1-\alpha_c)\,\widehat{i}_t \qquad (95)$$

$$\widehat{k}_{t+1}=\widehat{b}_t$$

$$\mathbf{K}_t = I_t + (1-\delta)K_t \rightarrow \widehat{k}_{t+1} = \delta\widehat{i}_t + (1-\delta)\,\widehat{k}_t$$

$$\begin{aligned}
w_t &= p_{mt}(1-\alpha)\frac{Y_t}{L_t} \rightarrow \widehat{w}_t = \widehat{p}_{mt} + \widehat{y}_t - \widehat{l}_t \\
1 + r_t^K &= \alpha p_{mt}\frac{Y_t}{K_t} + 1 - \delta \rightarrow \frac{r_t^K - r^K}{1+r^K} = \alpha_y\left(\widehat{p}_{mt} + \widehat{y}_t - \widehat{k}_t\right) && (96) \\
w_t &= Q_t^{1-\frac{\eta}{\sigma}} C_t^{\frac{1}{\sigma}} \psi H_t^{\varphi} \\
\widehat{w}_t &= \left(1 - \frac{\eta}{\sigma}\right)\widehat{q}_t + \frac{1}{\sigma}\widehat{c}_t + \varphi\widehat{l}_t \\
&= \frac{\sigma-\eta}{\sigma\eta}\left(\left(\alpha_{DD} + \alpha_{EE}\right)\left(\widehat{p}_t + \widehat{c}_t\right) - \begin{pmatrix} +\alpha_{DD}\beta_D\widehat{d}_t + \alpha_{EE}\beta_E\widehat{e}_t \\ +\left[\alpha_{DD}(1-\beta_D) + \alpha_{EE}(1-\beta_E)\right]\widehat{f}_t \end{pmatrix}\right) + \frac{1}{\sigma}\widehat{c}_t + \varphi\widehat{l}_t && (97)
\end{aligned}$$

where

$$\begin{aligned}
\alpha_c &\equiv \frac{C}{Y} \\
\alpha_y &\equiv \frac{\alpha\frac{\epsilon-1}{\epsilon}\frac{Y}{K}}{\alpha\frac{\epsilon-1}{\epsilon}\frac{Y}{K} + 1 - \delta}
\end{aligned}$$

Note that $r^K = i^K$ because inflation is zero in the SS.

We have also used labor market clearing condition $H_t = L_t$.

Summary of log-linearized equations with cash:

Define $\widetilde{x} = \widehat{x}_t - \widehat{p}_t$ for $x \in \{d, e, f, m\}$.

Parameters α_{FD}, β_J, α_m, α_c, α_y and α_{JJ} for $J \in \{D, E\}$ are all constants and defined in the appendix. Also define $\widehat{\pi}_t \equiv \Delta\hat{p}_t$. We have the following log-lin equations.

Here is the summary of log-linearized version of equilibrium conditions:

$$\text{Euler equation: } \widehat{c}_t = E_t\left[\widehat{c}_{t+1}\right] - \sigma\left(\beta i_t^S - E_t\left[\widehat{\pi}_{t+1}\right] + \beta - 1\right) + (\sigma-\eta)\left(E_t\left[\widehat{q}_{t+1}\right] - \widehat{q}_t\right) \quad (98)$$

$$\begin{aligned}
\text{CBDC demand:} \quad & \tfrac{i_t^S - i_t^F}{i^S - i^F} - 1 = \tfrac{1}{\eta}\widehat{c}_t - \alpha_{FD}\left(\tfrac{1}{\eta} - \tfrac{1}{\upsilon}\right)\beta_D\widetilde{d} - (1-\alpha_{FD})\left(\tfrac{1}{\eta} - \tfrac{1}{\varrho}\right)\beta_E\widetilde{e} \\
& - \left[\alpha_{FD}\left(\tfrac{1-\beta_D}{\eta} + \tfrac{\beta_D}{\upsilon}\right) + (1-\alpha_{FD})\left(\tfrac{1-\beta_E}{\eta} + \tfrac{\beta_E}{\varrho}\right)\right]\widetilde{f} && (99) \\
\text{Deposit demand:} \quad & \tfrac{i_t^S - i_t^D}{i^S - i^D} - 1 = \tfrac{1}{\eta}\widehat{c}_t - \left(\tfrac{1-\beta_D}{\upsilon} + \tfrac{\beta_D}{\eta}\right)\widetilde{d} - \left(-\tfrac{1}{\upsilon} + \tfrac{1}{\eta}\right)(1-\beta_D)\widetilde{f} && (100) \\
\text{Cash demand:} \quad & \tfrac{i_t^S}{i^S} - 1 = \tfrac{1}{\eta}\widehat{c}_t - \left(\tfrac{1-\beta_E}{\varrho} + \tfrac{\beta_E}{\eta}\right)\widetilde{e} - \left(-\tfrac{1}{\varrho} + \tfrac{1}{\eta}\right)(1-\beta_E)\widetilde{f} && (101)
\end{aligned}$$

Bank equations:

$$i_t^S - i_t^D = \ell^{-1}\left(i_t^S - i_t^M\right) \tag{102}$$

$$i_t^S - E_t r_{t+1}^K - \left(1 + r^K\right) E_t\left[\widehat{\pi}_{t+1}\right] = \rho\left(i_t^S - i_t^M\right) \tag{103}$$

$$\widetilde{d} = \alpha_m \widetilde{m} + (1 - \alpha_m)\,\widehat{b}_t \tag{104}$$

Philips curve:

$$\widehat{\pi}_t = \frac{(\epsilon - 1)\,Y}{\kappa}\widehat{p}_{mt} + \beta E_t\left[\widehat{\pi}_{t+1}\right] \tag{105}$$

The rest of the equations:

$$\widehat{y}_t = \alpha \widehat{k}_t + (1 - \alpha)\,\widehat{l}_t \tag{106}$$

$$\widehat{y}_t = \alpha_c \widehat{c}_t + (1 - \alpha_c)\,\widehat{i}_t \tag{107}$$

$$\widehat{k}_{t+1} = \widehat{b}_t \tag{108}$$

$$\widehat{k}_{t+1} = \delta \widehat{i}_t + (1 - \delta)\,\widehat{k}_t + \widehat{\xi}_{t+1} \tag{109}$$

$$\widehat{w}_t = \widehat{p}_{mt} + \widehat{y}_t - \widehat{l}_t \tag{110}$$

$$\frac{r_t^K - r^K}{1 + r^K} = \alpha_y\left(\widehat{p}_{mt} + \widehat{y}_t - \widehat{k}_t\right) + \widehat{\xi}_{t+1} \tag{111}$$

$$\widehat{w}_t = \left(1 - \frac{\eta}{\sigma}\right)\widehat{q}_t + \frac{1}{\sigma}\widehat{c}_t + \varphi \widehat{l}_t \tag{112}$$

$$\widehat{q}_t = \frac{1}{\eta}\left(\begin{array}{c} (\alpha_{DD} + \alpha_{EE})\,\widehat{c}_t \\ -\left(\begin{array}{c} +\alpha_{DD}\beta_D\widetilde{d} + \alpha_{EE}\beta_E\widetilde{e} \\ +\left[\alpha_{DD}(1-\beta_D) + \alpha_{EE}(1-\beta_E)\right]\widetilde{f}_t \end{array}\right)\end{array}\right) \tag{113}$$

D Special cases of the model

Now consider the general model with cash. Within this general model, we discuss some special cases that have been studied in the literature before: the no-CBDC case, a cash-like CBDC and a deposit-like CBDC. The goal is to show how this model can easily nest other models in the literature.

Special case: No CBDC

Here, we assume $\omega_{FD} = \omega_{FE} = 0$. The optimality conditions are modified to

$$\begin{aligned} \text{Deposits} \quad &: \quad \frac{i^S - i^D}{1 + i^S} = \omega_D \left(\frac{P_t C_t}{D_t} \right)^{\frac{1}{\eta}}, \\ \text{Cash} \quad &: \quad \frac{i^S}{1 + i^S} = \omega_E \left(\frac{P_t C_t}{E_t} \right)^{\frac{1}{\eta}}, \end{aligned}$$

where

$$\begin{aligned} Q_{D,t} &\equiv V_{D,t}^{-1} = \frac{D_t}{P_t C_t}, Q_{E,t} \equiv V_{E,t}^{-1} = \frac{E_t}{P_t C_t}, \\ Q_t &\equiv \left(1 + \omega_D V_{D,t}^{-(1-\frac{1}{\eta})} + \omega_E V_{E,t}^{-(1-\frac{1}{\eta})} \right)^{\frac{1}{1-\eta}}. \end{aligned}$$

This case nests Piazzesi et al.'s (2019) description of households. However, our description is still more general because not only deposits but also cash provides liquidity services here.

Special case: $\varrho = \upsilon = \eta$

In this case, agents have the love-of-variety feature in their means of payments, and the elasticity of consumption with respect to different means of payments are identical. The introduction of a CBDC here just enriches the set of means of payments that agents have available. The optimal conditions here imply that

$$\frac{i^S - i^D}{1 + i^S} \geq \omega_D \left(\frac{P_t C_t}{D_t} \right)^{\frac{1}{\eta}} \text{ with equality if } D_t > 0 \tag{114}$$

$$\frac{i^S}{1 + i^S} \geq \omega_E \left(\frac{P_t C_t}{E_t} \right)^{\frac{1}{\eta}} \text{ with equality if } E_t > 0 \tag{115}$$

$$\frac{i^S - i^F}{1 + i^S} \geq (\omega_{FD} + \omega_{FE}) \left(\frac{P_t C_t}{F_t} \right)^{\frac{1}{\eta}} \text{ with equality if } F_t > 0 \tag{116}$$

Here, the demand for different means of payments are not inter-related. The opportunity cost of each means of payment pins down the velocity and demand for that. This is true even if the utility function is not separable. The effect of non-separability will be reflected in the labor supply equation.

CBDC is a perfect substitute for deposits: $\upsilon = \infty$

Here, we study a CBDC that is a perfect substitute for bank deposits, i.e, $\upsilon = \infty$. It is similar to a deposit-like CBDC which has been discussed in the literature, but also offers some degree of substitution with cash.[17] In this case, $Q_{D,t}$ is modified to $Q_{D,t} = V_{D,t}^{-1} + \frac{\omega_{FD}}{\omega_D} V_{F,t}^{-1} =$

[17] A solely deposit-like CBDC would require $\omega_{FE} = 0$.

$\frac{D_t+\frac{\omega_{FD}}{\omega_D}F_t}{P_tC_t}$. Therefore, the optimality conditions imply

$$\text{Deposits:} \qquad \frac{i^S - i^D}{1+i^S} = \omega_D \left(\frac{P_tC_t}{D_t + \frac{\omega_{FD}}{\omega_D}F_t} \right)^{\frac{1}{\eta}}$$

$$\text{Cash:} \qquad \frac{i^S - i^F}{1+i^S} = \omega_{FD} \left(\frac{P_tC_t}{D_t + \frac{\omega_{FD}}{\omega_D}F_t} \right)^{\frac{1}{\eta}} + \omega_{FE} \left(\frac{P_tC_t}{F_t} \right)^{\frac{1}{\varrho}} Q_{E,t}^{\frac{1}{\varrho}-\frac{1}{\eta}}$$

assuming that the CBDC is used in equilibrium. One can divide the second optimality condition by the first to obtain an equation for the opportunity cost of holding CBDC relative to that of deposits. For simplicity, assume $\varrho = \eta$, then we have

$$\frac{i^D - i^F}{i^S - i^D} = \frac{\omega_{FD} - \omega_D}{\omega_D} + \frac{\omega_{FE}}{\omega_D} \left(\frac{D_t}{F_t} + \frac{\omega_{FD}}{\omega_D} \right)^{\frac{1}{\eta}}.$$

This equation implies that the wedge between the interest rate of this type of CBDC and deposits depends on the relative usefulness of CBDC in deposit transactions as well as the liquidity service it provides in cash transactions. This equation is especially useful because it is a function of the relative quantity of deposits and CBDC and does not depend on the quantity of cash used in transactions (which is a direct implication of $\varrho = \eta$).

Finally, in a special case where CBDC can be used in exactly the same set of transactions as deposits with the same importance, $\omega_{FD} = \omega_D$, we will have $i^D \geq i^F$. The central bank pays less interest on CBDC compared with bank deposits as long as CBDC provides liquidity not only to deposit transactions but also in some transactions where cash can currently be used.

CBDC is a perfect substitute for cash: $\varrho = \infty$
Here we study a CBDC that is a perfect substitute for cash, i.e, $\varrho = \infty$. It is similar to the cash-like CBDC discussed in the literature but also offers some degree of substitution with deposits. In this case, the optimality conditions imply:

$$\text{Cash:} \qquad \frac{i^S}{1+i^S} \geq \omega_E Q_{E,t}^{-\frac{1}{\eta}} \text{ with eq if } E_t > 0,$$

$$\text{CBDC:} \qquad \frac{i^S - i^F}{1+i^S} \geq \omega_{FD} \left(\frac{P_tC_t}{F_t} \right)^{\frac{1}{\upsilon}} Q_{D,t}^{\frac{1}{\upsilon}-\frac{1}{\eta}} + \omega_{FE} Q_{E,t}^{-\frac{1}{\eta}} \text{ with eq if } F_t > 0.$$

If cash and CBDC are both used in equilibrium, then:

$$\frac{-i^F}{1+i^S} = (\omega_{FE} - \omega_E)\, Q_{E,t}^{-\frac{1}{\eta}} + \omega_{FD} \left(\frac{P_tC_t}{F_t} \right)^{\frac{1}{\upsilon}} Q_{D,t}^{\frac{1}{\upsilon}-\frac{1}{\eta}}.$$

Note, for example, that when $\omega_{FE} = \omega_E$, a positive interest on CBDC ($i^F \geq 0$) implies that cash will be out of circulation as it is strictly dominated by CBDC. In general, the higher the difference $\omega_{FE} - \omega_E$ or the higher the usefulness of CBDC in other transactions (higher ω_{FD}), the lower the CBDC rate can go.

Now we analyze the **steady state of various special cases**.

The zero inflation rate steady state with an endogenous quantity of reserves

Special case when $\eta = \sigma = 1$. As i^M goes up, i^K goes up too.

We first assume $\frac{\eta}{\sigma} = 1$ and $\delta = 0$, for simplicity. An increase in i^M **makes loans more expensive for firms, so output goes down**. However, notice that because $\frac{\eta}{\sigma} = 1$, the payment side does not have any effects on the opportunity cost of lending and on deposit rates because they are both determined only by cost of reserves.

Notice that in this case, if CBDC and deposits are substitutes to some extent, as i^F goes down, then D goes down too, and at some point, D/P goes less than $\ell \rho b$, at which point $M \geq 0$ will be binding. This means that banks cannot attract enough deposits to raise resources for their lending.

Altogether, when $\frac{\eta}{\sigma} = 1$, the disintermediation channel does not operate in this model and the lending side is separated from the deposit side given that the reserve requirement is binding and the interest on CBDC is low enough.

Result: When $\sigma = \eta$, the output depends only on interest on reserves, i^M (and not on the interest on CBDC i^F), and i^F only determines the quantity of real balances demanded.

Special case when $\varrho = \upsilon = \eta$. In this special case, we can calculate Q in a closed form. Rewrite FOCs:

$$\text{Deposit demand:} \qquad \frac{i^S - i^D}{1 + i^S} = \omega_D V_D^{\frac{1}{\eta}} \tag{117}$$

$$\text{Cash demand:} \qquad \frac{i^S - i^E}{1 + i^S} = \omega_E V_E^{\frac{1}{\eta}} \tag{118}$$

$$\text{CBDC demand:} \qquad \frac{i^S - i^F}{1 + i^S} = \omega_F V_F^{\frac{1}{\eta}} \tag{119}$$

where $\omega_F \equiv \omega_{FD} + \omega_{FE}$. Hence, Q is given by

$$\begin{aligned} Q &\equiv \left(1 + \omega_D V_{FD}^{\frac{1}{\eta}-1} + \omega_E V_{FE}^{\frac{1}{\eta}-1}\right)^{\frac{1}{1-\eta}} \\ &= \left(1 + \omega_D V_D^{\frac{1}{\eta}-1} + \omega_E V_E^{\frac{1}{\eta}-1} + \omega_F V_F^{\frac{1}{\eta}-1}\right)^{\frac{1}{1-\eta}} \\ &= \left(1 + \sum_J \omega_J^{\eta} \left(\frac{i^S - i^J}{1 + i^S}\right)^{1-\eta}\right)^{\frac{1}{1-\eta}} \end{aligned} \tag{120}$$

Result:

- When $\sigma \neq \eta$, the output depends on policy from two channels. First, Y is a function of i^K directly and i^K is determined by i^M. Second, Y is a function of Q, which depends on the CBDC rate (as a decreasing function) and also depends on i^D, which is again determined by i^M.

More specifically, when $\sigma > \eta$, there is complementarity between consumption and other means of payments. Paying more interest on CBDC decreases Q and increases Y.

Note that in this case ($\varrho = \upsilon = \eta$), we do not see the disintermediation channel because CBDC is not a perfect substitute for deposits.

Another special case: CBDC and deposits are perfect substitutes: $\upsilon = \infty$, $\omega_D = \omega_{FD}$ **and** $\omega_{FE} = 0$

$$\text{Deposit and CBDC demand:} \quad \frac{i^S - i^D}{1+i^S} = \omega_D V_{FD}^{\frac{1}{\eta}} = \frac{i^S - i^F}{1+i^S} \tag{121}$$

$$\text{Cash demand:} \quad \frac{i^S - i^E}{1+i^S} = \omega_E V_E^{\frac{1}{\varrho}} V_{FE}^{-\frac{1}{\varrho}+\frac{1}{\eta}} \tag{122}$$

$$\begin{aligned} V_{FD} &\equiv \left(V_D^{-1} + V_F^{-1}\right)^{-1} = \frac{PC}{D+F} \\ V_{FE} &\equiv V_E \\ Q &\equiv \left(1 + \omega_D V_{FD}^{\frac{1}{\eta}-1} + \omega_E V_E^{\frac{1}{\eta}-1}\right)^{\frac{1}{1-\eta}} \end{aligned}$$

In this case, interest on deposits is pinned down by i^M, on the one hand. On the other hand, it is determined by the CBDC interest rate.

More specifically, for a given i^M:

- If $i^F > i^D$, bank cannot raise deposits. Therefore, $i^D \geq i^F$.
- If $i^F < i^D$, demand for CBDC is zero and CBDC is not used.
- If $i^D = i^F$, then agents are indifferent between CBDC and deposits.

For a given i^F, the maximum deposit demand is given by

$$\frac{PC}{V_{FD}} = \frac{PC}{\left(\frac{i^S - i^F}{\omega_D(1+i^S)}\right)^{\eta}}.$$

- If i^M implies an i^D strictly lower than i^F, then banks cannot raise deposits, implying that production cannot take place, which is not possible. This means our initial assumption that banks hold reserves is violated. In this case, banks increase their rate to i^F to compete with CBDC and be able to raise deposits. They don't invest in reserves because their rate is too low. (They would have borrowed reserves if we had allowed them). In this case, $\frac{i^S - i^D}{1+i^S} = \frac{i^S - i^F}{1+i^S} = \frac{i^S - i^K}{(1+i^S)\rho\ell} < \frac{i^S - i^M}{(1+i^S)\ell}$. The CBDC interest rate is the floor for deposit rates. (This case is close to Keister and Sanches (2023)).

- In the knife-edge case, i^M implies an i^D exactly equal to i^F. In this case, agents are indifferent between the CBDC and deposits.
- If i^M implies an i^D higher than i^F, then the CBDC is not used.

It is easy to incorporate market power into the model. In that case, we can compare our results with Chiu et al. (2023).

The zero inflation rate steady state with a fixed quantity of reserves

The only difference is that here i^M is endogenous and M/P (real supply of reserves) is exogenous and set by the policy.

Endogenous variables:

- Price level, output and consumption and labor: Y, C, L
- Nominal balances: D, E
- Real assets: b
- Rates: i^M, i^K, i^D

Policy tools: $M/P, i^F$.

We follow the same three steps as before. However, since we do not know i^M, we have to start with a guess for i^M and then solve for a fixed point. More specifically:

- Start with a guess for i^M and derive the value for M/P (demand for reserves) by solving the three blocks in the previous subsection. That is, for the given i^M, start from Block 1 and follow the same steps summarized in Equations (14) to (21) and derive M/P from (21). Call it m_0, which is the demand for reserves.
- If m_0 is higher than the exogenous M/P set by the policy, we have to decrease i^M in the next iteration; otherwise, we have to increase i^M.
- Continue this until we converge and get the value for i^M.

Note that a higher i^M means that banks receive more benefits for reserves, so their demand for reserves goes up.

Analytical derivation of Q for the case with cash

We **assume** $\varrho = \upsilon$ so that we can simplify the analytical derivation of Q as much as possible:

$$\text{Deposit demand:} \qquad \frac{i^S - i^D}{1+i^S} = \omega_D V_D^{\frac{1}{\upsilon}} Q_D^{-\frac{1}{\upsilon}+\frac{1}{\eta}}, \tag{123}$$

$$\text{Cash demand:} \qquad \frac{i^S - i^E}{1+i^S} = \omega_E V_E^{\frac{1}{\upsilon}} Q_E^{-\frac{1}{\upsilon}+\frac{1}{\eta}}, \tag{124}$$

These imply that

$$V_D^{\frac{1-\upsilon}{\upsilon}} = \omega_D^{\upsilon-1}\left(\frac{i^S-i^D}{1+i^S}\right)^{1-\upsilon} Q_D^{\frac{\upsilon-\eta}{\eta}\frac{1-\upsilon}{\upsilon}},$$
$$V_E^{\frac{1-\upsilon}{\upsilon}} = \omega_E^{\upsilon-1}\left(\frac{i^S-i^E}{1+i^S}\right)^{1-\upsilon} Q_E^{\frac{\upsilon-\eta}{\eta}\frac{1-\upsilon}{\upsilon}},$$

or equivalently,

$$V_D^{\frac{-1}{\upsilon}} = \omega_D\left(\frac{i^S-i^D}{1+i^S}\right)^{-1} Q_D^{\frac{\upsilon-\eta}{\eta}\frac{-1}{\upsilon}},$$
$$V_E^{\frac{-1}{\upsilon}} = \omega_E\left(\frac{i^S-i^E}{1+i^S}\right)^{-1} Q_E^{\frac{\upsilon-\eta}{\eta}\frac{-1}{\upsilon}}.$$

Remember we had

$$Q_{D,t}^{1-\frac{1}{\upsilon}} \equiv V_{D,t}^{\frac{1}{\upsilon}-1} + \frac{\omega_{FD}}{\omega_D}V_{F,t}^{\frac{1}{\upsilon}-1},$$
$$Q_{E,t}^{1-\frac{1}{\upsilon}} \equiv V_{E,t}^{\frac{1}{\upsilon}-1} + \frac{\omega_{FE}}{\omega_E}V_{F,t}^{\frac{1}{\upsilon}-1}.$$

CBDC demand:

$$\text{CBDC demand:}\frac{i^S-i^F}{1+i^S} = \omega_{FD}V_F^{\frac{1}{\upsilon}}Q_D^{-\frac{1}{\upsilon}+\frac{1}{\eta}} + \omega_{FE}V_F^{\frac{1}{\upsilon}}Q_E^{-\frac{1}{\upsilon}+\frac{1}{\eta}},$$
$$\Rightarrow i^S-i^F = \frac{\omega_{FD}}{\omega_D}\left(\frac{V_D}{V_F}\right)^{\frac{-1}{\upsilon}}\left(i^S-i^D\right) + \frac{\omega_{FE}}{\omega_E}\left(\frac{V_E}{V_F}\right)^{\frac{-1}{\upsilon}} i^S.$$
$$\Rightarrow V_F^{\frac{-1}{\upsilon}} = \left[\omega_{FD}Q_D^{\frac{\upsilon-\eta}{\eta}\frac{-1}{\upsilon}} + \omega_{FE}Q_E^{\frac{\upsilon-\eta}{\eta}\frac{-1}{\upsilon}}\right]\left(\frac{1+i^S}{i^S-i^F}\right)$$
$$\Rightarrow V_F^{\frac{1-\upsilon}{\upsilon}} = \left[\omega_{FD}Q_D^{\frac{\upsilon-\eta}{\eta}\frac{-1}{\upsilon}} + \omega_{FE}Q_E^{\frac{\upsilon-\eta}{\eta}\frac{-1}{\upsilon}}\right]^{\upsilon-1}\left(\frac{1+i^S}{i^S-i^F}\right)^{\upsilon-1}$$

Hence, we have

$$Q_D^{1-\frac{1}{\upsilon}} \equiv \frac{\omega_D^{\upsilon}}{\omega_D}Q_D^{\frac{\upsilon-\eta}{\eta}\frac{1-\upsilon}{\upsilon}}\left(\frac{i^S-i^D}{1+i^S}\right)^{1-\upsilon} + \frac{\omega_{FD}}{\omega_D}\left[\omega_{FD}Q_D^{\frac{\upsilon-\eta}{\eta}\frac{-1}{\upsilon}} + \omega_{FE}Q_E^{\frac{\upsilon-\eta}{\eta}\frac{-1}{\upsilon}}\right]^{\upsilon-1}\left(\frac{i^S-i^F}{1+i^S}\right)^{1-\upsilon}, \quad (125)$$
$$Q_E^{1-\frac{1}{\upsilon}} \equiv \frac{\omega_E^{\upsilon}}{\omega_E}Q_E^{\frac{\upsilon-\eta}{\eta}\frac{1-\upsilon}{\upsilon}}\left(\frac{i^S-i^E}{1+i^S}\right)^{1-\upsilon} + \frac{\omega_{FE}}{\omega_E}\left[\omega_{FD}Q_D^{\frac{\upsilon-\eta}{\eta}\frac{-1}{\upsilon}} + \omega_{FE}Q_E^{\frac{\upsilon-\eta}{\eta}\frac{-1}{\upsilon}}\right]^{\upsilon-1}\left(\frac{i^S-i^F}{1+i^S}\right)^{1-\upsilon}, \quad (126)$$

These two equations give us two unknowns. We can then put them into the following equation to derive Q:

$$Q_t \equiv \left(1+\omega_D Q_{D,t}^{1-\frac{1}{\eta}} + \omega_E Q_{E,t}^{1-\frac{1}{\eta}}\right)^{\frac{1}{1-\eta}}.$$