BANK OF CANADA
BANQUE DU CANADA

Working Paper/Document de travail
2011-4

Counterfeit Quality and Verification in a Monetary Exchange

by Ben S. C. Fung and Enchuan Shao

www.bank-banque-canada.ca

Bank of Canada Working Paper 2011-4

February 2011

Counterfeit Quality and Verification in a Monetary Exchange

by

Ben S. C. Fung and Enchuan Shao

Currency Department
Bank of Canada
Ottawa, Ontario, Canada K1A 0G9
bfung@bankofcanada.ca
eshao@bankofcanada.ca

ISSN 1701-9397

Acknowledgements

The authors would like to thank Jonathan Chiu, Mei Dong, Yiting Li, Guillaume Rocheteau, Shouyong Shi, Steve Williamson, Randy Wright, and seminar participants at the Bank of Canada, Canadian Economic Associations Annual Conferences, Federal Reserve Bank of Chicago Summer Money Macro Workshop, Midwest Macroeconomic Meetings, Workshop in Tsinghua University, University of Iowa for their comments and useful discussions. The authors alone are responsible for all errors and omissions.

Abstract

Recent studies on counterfeiting in a monetary search framework show that counterfeiting does not occur in a monetary equilibrium. These findings are inconsistent with the observation that counterfeiting of bank notes has been a serious problem in some countries. In this paper, we show that counterfeiting can exist as an equilibrium outcome in a model in which money is not perfectly recognizable and thus can be counterfeited. A competitive search environment is employed in which sellers post offers and buyers direct their search based on posted offers. When sellers are uninformed about the quality of the money, their offers are pooling and thus buyers can extract rents by using counterfeit money. In this case, counterfeit notes can coexist with genuine notes under certain conditions. We also explicitly model the interaction between sellers' verification decisions and counterfeiters' choices of counterfeit quality. This allows us to better understand how policies can affect counterfeiting.

JEL classification: D82, D83, E42, E50
Bank classification: Bank notes

Résumé

De récentes études réalisées à partir d'un cadre monétaire de prospection montrent que la contrefaçon de billets n'existe pas en situation d'équilibre monétaire. Cette conclusion va à l'encontre de ce qui s'observe dans certains pays où la contrefaçon a été un problème de taille. Les auteurs montrent que le faux-monnayage peut faire partie de l'équilibre d'un modèle à l'intérieur duquel les billets, faute d'être parfaitement identifiables, risquent d'être contrefaits. Le modèle utilisé s'appuie sur un cadre de prospection concurrentielle dans lequel les vendeurs publient des offres qui aiguillent les acheteurs. Quand les vendeurs ne connaissent pas la qualité des billets, ils ne différencient pas leur offre selon les acheteurs, ce qui permet à ces derniers de tirer un bénéfice de l'emploi de faux billets. Dans ce cas de figure, la coexistence des billets contrefaits et des billets authentiques est possible à certaines conditions. Les auteurs formalisent par ailleurs en détail la relation entre la décision des vendeurs de vérifier l'authenticité des billets et le choix de la qualité des contrefaçons opéré par les faussaires. Cette démarche permet de mieux comprendre comment les politiques parviennent à influer sur la contrefaçon.

Classification JEL : D82, D83, E42, E50
Classification de la Banque : Billets de banque

1 Introduction

As the sole provider of currency in Canada, the Bank of Canada aims to supply bank notes that Canadians can use with confidence. In the early 2000s, however, there was a sharp increase in counterfeit bank notes in Canada, rising to a peak of almost 500 counterfeit notes detected per million notes in circulation (PPM) in 2004 from well below 100 for most of the 1990s (see Figure 1). This increase in counterfeiting could threaten the public's confidence in bank notes. In response, the Bank of Canada focused its efforts on developing bank notes that are difficult to counterfeit, promoting the deterrence of counterfeiting by law enforcement and prosecutors, promoting the routine verification of bank notes by retailers, and withdrawing worn notes and more vulnerable older-series notes from circulation. As a result, counterfeiting has declined considerably since 2004 and is now back down to the pre-2000 level.[1] Figure 2 shows that counterfeiting is also a problem in some countries such as the United Kingdom and Mexico but not at all a problem in others such as Australia and Korea.

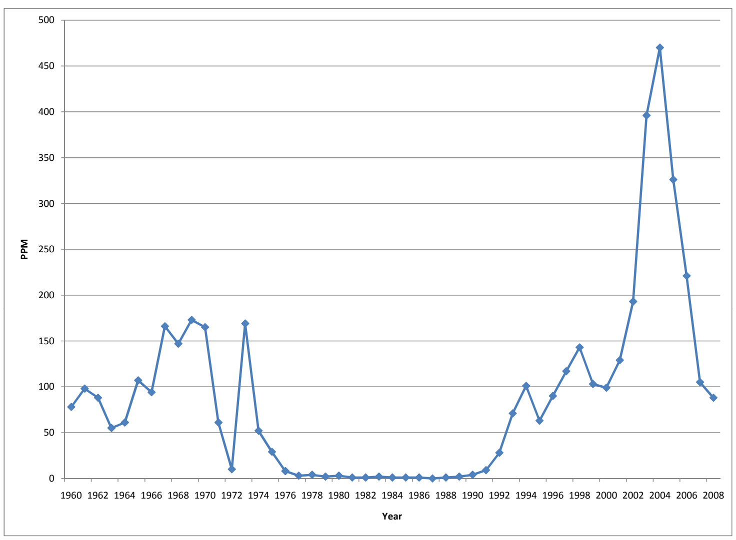

Figure 1: Counterfeiting Activities in Canada since 1960

What have caused the rise in counterfeiting and the subsequent decline? Why have some countries experienced a more serious counterfeiting problem than others? What measures are effective in reducing counterfeiting? Given that anti-counterfeiting measures are costly to implement, how important is it to continue to invest in these measures when counterfeiting has subsided? Do macroeconomic variables such as

[1] For a brief discussion of the Canadian experienc of counterfeiting, see for example Bank of Canada (2008).

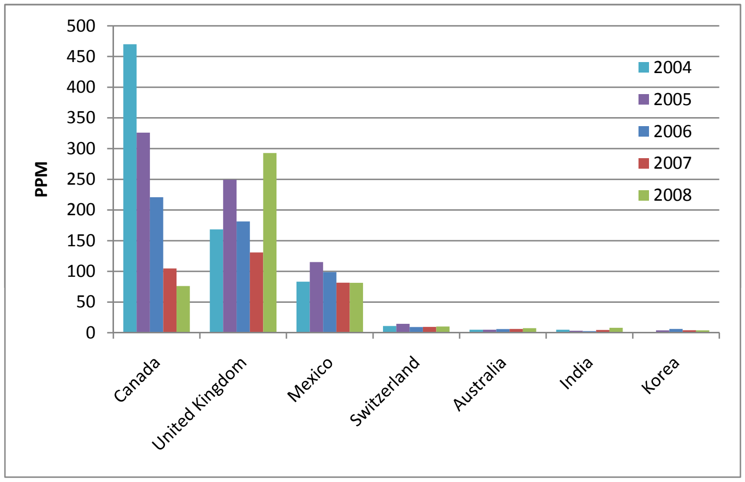

Figure 2: No. of counterfeit notes detected in selected countries

inflation have any effects on counterfeiting? How costly is counterfeiting to society? These questions are of interest to policy makers and academics. In order to address these questions, one approach is to develop a monetary model in which counterfeiting arises endogenously. Such a model should incorporate relevant decisions such as counterfeiting, verification, and government policy. This paper contributes to such an approach by studying a search model of money in which money is not perfectly recognizable and thus can be counterfeited. We show that counterfeiting can exist as an equilibrium outcome and characterize the conditions under which such an equilibrium will exist. In addition, we would like to examine how policies will affect counterfeiting.[2]

The literature on theoretical models of counterfeiting of bank notes is relatively small. In recent years, a few papers have studied counterfeiting in the context of monetary search models.[3] For example, Kultti (1996) studies the conditions under which a monetary equilibrium can be sustained by extending the search models of Kiyotaki and Wright (1993) while Green and Weber (1996) look at how the introduction of new issue of bank notes affect counterfeiting. Cavalcanti and Nosal (2007) argue that

[2] For example, Green and Weber (1996) examine whether a policy of introducing a new style of currency that is harder to counterfeit but not immediately to withdraw from circulation all of the old issues would be able to reduce counterfeiting. Monnet (2005) studies whether inflation would reduce the value of counterfeiting activities.

[3] Another strand of literature studies counterfeiting using game-theoretical models, for example, Lengwiler (1997) and Quercioli and Smith (2007).

it is optimal to tolerate counterfeiting when transactions are difficult to mornitor and their values are small. Nosal and Wallace (2007) show that counterfeiting is only a threat and does not exist in a monetary equilibrium. However, such a threat could potentially result in the collapse of a monetary equilibrium if the cost of counterfeiting is sufficiently low. Li and Rocheteau (2008), with a basic set up similar to Nosal and Wallace, argue that despite the threat of counterfeiting there always exists a monetary equilibrium. However, the threat of counterfeiting will instead affect real allocations and thus social welfare. In the base model, Li and Rocheteau also find no monetary equilibrium with counterfeiting.[4] However, the experiences in Canada discussed above and in other countires suggest that counterfeiting is more than just a threat. Indeed, many countries have experienced a rapid increase in counterfeiting of bank notes followed by a gradual decline over the last decade or so. To explain such a phenomenon, it requires a model in which counterfeiting of bank notes exists as an equilibrium outcome.

Our model differs from existing models of counterfeiting in several aspects. First, money is divisible as in Lagos and Wright (2005). In Kultti (1996), Cavalcanti and Nosal (2007) and Williamson (2002), however, money is indivisible and thus counterfeit notes can improve welfare by acting as private money in alleviating the money shortage problem. In Canada and other industrialized countries, however, money shortage is less likely to be an issue. Second, the market structure in our model is such that sellers post their offers to buyers and thus buyers can direct their search based on the posted offers.[5] Unlike Nosal and Wallace, and Li and Rocheteau, buyers cannot signal to the sellers their types. In this case, there will always be a monetary equilibrium. Third, buyers have to pay a cost to produce counterfeit notes, and in the extended version of our model a higher quality counterfeit note is more costly to produce. In turn, higher quality counterfeit notes are more difficult for the seller to detect. In addition, sellers can invest in a verification technology that can detect counterfeit notes with a probability. If a seller does not invest in the technology, she will not be able to tell between genuine and counterfeit notes. The seller's decision may or may not be known to the buyer. Thus it allows us to explicitly model the interaction between counterfeiters and sellers.

We begin with a baseline model of counterfeiting which is very similar to Nosal and Wallace (2007), except that we consider divisible money and competitive search. Buyers decide whether to produce counterfeits or not at a cost. Sellers will receive a signal which will inform them whether the notes they will receive are genuine or counterfeit at some positive probability. We then characterize the conditions under which a monetary equilibrium with counterfeiting will exist in such an environment. We find that counterfeiting can exist in a monetary equilibrium if the cost of producing

[4]Li and Rocheteau consider two extensions in which counterfeiting can exist in equilbrium.

[5]Competitive search is also a more realistic description of most transactions at the retail level. A buyer usually can observe the price listed for the goods or services she wants to buy and then decides which store to go to.

counterfeits is sufficiently low. Next we consider an extension to the baseline model in which a counterfeiter decides on counterfeit quality and a seller decides whether to verify the notes they receive or not. Such decisions will influence the probability that the signal is informative. A buyer's decision regarding counterfeiting is always private information. However, a seller's verification choice may or may not be observable. In these cases, the conditions for the existence of counterfeiting in equilibrium are related to the money growth rate and the cost of verification. We also find that a higher rate of inflation tends to reduce counterfeiting. This is consistent with the observation that counterfeiting is less likely to be a serious problem in high inflation countries and that countries experencing a high level of counterfeiting, such as Canada and the United Kingdom, have relatively low and stable inflation. Interestingly, we find that a higher cost of verification tends to reduce counterfeiting. This seems counterintuitive. The reason for this result is that when the cost of verification is higher, a seller will enter the market only if there is a higher fraction of buyers using genuine money so that he can make enough money by selling to a buyer.

The rest of the paper is organized as follows. The next section describes the model environment. In Section 3, we consider the baseline mode of counterfeiting in a competitive search environment and derive conditions under which counterfeiting can exist in a monetary equilibrium. In Section 4, we consider an extension which include decisions regarding counterfeit quaility and verification. We first cosider the case that the seller's verification decision is public information. We again derive conditions under which counterfeiting can exist in a monetary equilibrium and then study how changes in inflation and the cost of verification will affect counterfeiting and the quantities traded. In Section 5, we allow the seller's verification decision to be private information and study whether the results in the previous section will be affected. Section 6 concludes.

2 The Environment

The basic economic environment is similar to Rocheteau and Wright (2005). Time is discrete and runs forever. Each period is divided into two sub-periods, day and night, during which the market structure differs. During the day, there is a Walrasian market characterized by competitive trading, while at night there is a search market characterized by bilateral trading. There is a continuum of infinitely-lived agents who differ across two dimensions. First, they have private information on some of their own characteristics that will be described in detail later. Second, they belong to one of two groups in the search market, called *buyers* and *sellers*. We normalize the measure of buyers to 1. In the Walrasian market all agents produce and consume but in the search market a buyer can only consume and a seller can only produce. This specification on agents' trading roles in the search market generates a lack-of-double-coincidence-of-wants problem. Therefore, barter is ruled out. All meetings are assumed to be anonymous which precludes credit. These frictions make a medium of exchange essential in the search market.

Goods are perishable while (genuine) fiat money is storable and thus money can potentially be used as a medium of exchange. Money is perfectly divisible and its stock at time t is given by M_t . The money stock grows at a constrant gross rate γ, so that $M_{t+1} = \gamma M_t$. New money is injected ($\gamma > 1$) or withdrawn ($\gamma < 1$) via lump sump transfers to all agents in the Walrasian market. We restrict attention to policies where $\gamma \geq \beta$, where $\beta \in (0, 1)$ is the discount factor, since it is easy to check that there is no equilibrium otherwise. To examine what happens when $\gamma = \beta$, which is the Friedman rule, we can take the limit of equilibria as $\gamma \longrightarrow \beta$.

Money is perfectly recognizable in the Walrasian market but imperfectly recognizable in the search market. The recognizability problem of fiat money gives a buyer an incentive to produce counterfeits and extract more surplus in the bilateral trade. Buyers can produce counterfeited notes in any quantity at a cost and this decision is private information. In any trade meeting, the trading pair will receive a signal regarding the quality of the money used by the buyer. With probability π, ths signal reveals the type of money used by the buyer and with probabiliy $1 - \pi$, the signal is uninformative. We will consider two different cases regarding this signal. In the first case, the baseline model, the probabily of this signal being informative is exogenous, as in Nosal and Wallace (2007). This case is important because no counterfeit equilibrium is found when the buyer makes a take-it-or-leave-it offer to the seller. It is thus of interest to study whether counterfeiting can exist in equilibrium under a different trading mechanism such as competitive search. In the second case, the probability depends on the actions of the counterfeiters and the sellers. More specifically, counterfeiters can choose a level of the quality of counterfeit notes that affect π while sellers can choose their verification effort that will influence π. In this case, we can study the inteaction between the counterfeiter's decision regarding quality and the seller's decision regarding verification. As such, we can assess how anti-counterfeiting policies will affect counterfeiting. We will describe the second case in more detail below.

The counterfeiters can choose the quality level of the counterfeit notes produced. Higher quality notes are more costly to produce but they are less likely to be detected by the seller than lower quality ones. Let $h \in [0, +\infty)$ denote the quality level of counterfeits chosen by a buyer. Thus the probability π is a function of the quality h such that $\pi = \pi(h)$ and $\pi'(h) < 0$. Denote $g\,(h)$ the cost that a buyer pays to produce counterfeits of quality h. Assume that $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is an increasing and convex function and satisfies $g\,(0) = 0$. It is important to note that the cost of producing counterfeits depends on the quality but not the quantity produced. Counterfeits are assumed to be 100% disintegrated or confiscated at the end of each period as in Nosal and Wallace (2007). As a result, there is no incentive for sellers to produce or pass counterfeits.

A seller can choose to verify a buyer's money holding in a bilateral meeting by paying a fixed utility cost L to invest in a verification technology or to exert a verification effort. In this case, $\pi > 0$, and the public signal reveals the quality of a buyer's holding of money according to the following probability[6]:

$$\begin{cases} \pi(h) & \text{if the buyer produces counterfeits,} \\ \pi(H) & \text{otherwise.} \end{cases}$$

With the complementary probability $1 - \pi$ the seller is uninformed. Here H is the economy-wide counterfeit quality. In equilibrium $h = H$. If the seller chooses not to verify, he will receive no information about the type of money held by the buyer, i.e. $\pi = 0$ regardless of the quality of the counterfeit notes. In other words, a seller that does not verify will be completely uninformed regarding whether the notes are counterfeit or genuine. In what follows, we describe the environment based on the case where the probability π depends on the counterfeiter's choice on counterfeit quality and the seller's decision on verification.

Let $\varepsilon \in \{0, 1\}$ denote the public signal that a seller receives, where $\varepsilon = 1$ indicates that the seller is informed and $\varepsilon = 0$ indicates the opposite. Notice that detection is always imperfect and the detection probability depends on the quality of counterfeits but not the quantity. The function $\pi : \mathbb{R}^{+} \to [0, 1]$ is decreasing and convex. Let $a \in \{y, n\}$ be the action that the seller will take where $a = y$ means that the seller chooses to verify while $a = n$ implies he does not. The seller's action may or may not be observed by all other agents.[7] We will first consider the case where a is observable, and then later we will relax this assumption to allow a to be private information. The corresponding cost with respect to a is therefore

$$f(a) = \begin{cases} L, & \text{if } a = y \\ 0, & \text{if } a = n \end{cases}.$$

The instantaneous utility of a seller at date t is

$$U_t^s = v(x_t) - y_t - q_t - f(a_t),$$

[6]There are two detection probabilities contingent on whether a buyer produces counterfeits or not. If a buyer produces counterfeits, the seller's detection probability will depend on the individual's quality level of counterfeits. In the case of a buyer who holds only genuine money, however, the detection probability cannot depend on the individual's quality level of counterfeits since the buyer does not produce counterfeits. To be consistent with the assumption of imperfect recognizability of money, it is natural to assume that the detection probability is related to economy-wide counterfeit quality when a seller meets with a genuine money holder. Of course, in equilibrium these two probabilities are equal.

[7]Sometimes, retailers will install a device such as an ultra violet light to verify the bank notes and place them in a location where they are visible to buyers. Or a buyer may observe a cashier in a store verifying notes received in a transaction. In these cases, the buyer knows that the seller will likely verify. In other cases, the buyer does not know whether the seller will verify either because the buyer cannot observe how other transactions are done or because the seller does not know if the seller has installed any machine that helps verification.

where x_t is the quantity consumed and y_t is the quantity produced during the day, q_t is production at night. Lifetime utility for a seller is $\sum_{t=0}^{\infty} \beta^t U_t^s$. We assume that $v'(x) > 0$, $v''(x) < 0$ for all x, and there exists $x^* > 0$ such that $v'(x^*) = 1$. Similarly, the instantaneous utility of a buyer is

$$U_t^b = v(x_t) - y_t + u(q_t) - g(h_t)\,\mathbb{I}_{\{i=c\}},$$

where q_t is the quantity consumed at night and $\mathbb{I}$ is the indicator function which is equal to 1 if the buyer chooses to produce counterfeits ($i = c$) or 0 if the buyer holds only genuine money ($i = g$). Lifetime utility for a buyer is $\sum_{t=0}^{\infty} \beta^t U_t^b$. Assume that $u(0) = 0$, $u'(0) = +\infty$, $u'(q) > 0$, and $u''(q) < 0$. There exists $\overline{q}$ and $q^* > 0$ such that $u(\overline{q}) = \overline{q}$ and $u'(q^*) = 1$.

The terms of trade are determined in the competitive search market in the spirit of Moen (1997) and Rocheteau and Wright (2005). Each seller decides whether to incur a cost to verify before entering the search market. Prior to the search process, each seller simultaneously posts an offer that specifies the terms at which buyers and sellers commit to trade. Specifically, an offer is a schedule $\{q_\varepsilon^i, d_\varepsilon^i\}$ specifying the quantity traded q_ε^i and the total monetary payment d_ε^i conditional on the realization of the verification signal ε and the buyer's type i.[8] (In principle, the offers can be different across different types of sellers. We will discuss offers in detail when we describe the equilibrium concepts.) Buyers then observe all the posted offers and direct their search towards those sellers posting the most attractive offer. The set of sellers posting the same offer and the set of buyers directing their search towards them form a sub-market. In each sub-market, buyers and sellers meet randomly according to the matching function discussed below. When a buyer and a seller meet, the buyer decides to either accept the offer and commit to the terms it specifies, or abandon all trade. If the offer is accepted, the signal is realized and the buyer purchases q_ε^i units for d_ε^i dollars.

The probability that a buyer and a seller are paired off is independent of the buyer's or seller's type. To focus on the informational frictions and to avoid unnecessary complications, we assume that individuals experience at most one match and that matching is efficient, meaning that the short-side of the market is always served in each sub-market. The probability that a buyer matches a seller is then

$$\alpha^b(\theta_j) = \min\left(1, \theta_j^{-1}\right), \tag{1}$$

where θ_j is the ratio of buyers over sellers in sub-market j, or the market tightness. Similarly, the probability that a seller matches a buyer is

$$\alpha^s(\theta_j) = \min(1, \theta_j). \tag{2}$$

[8]The offer does not include the amount of counterfeits because counterfeits have no value and sellers are not willing to accept them.

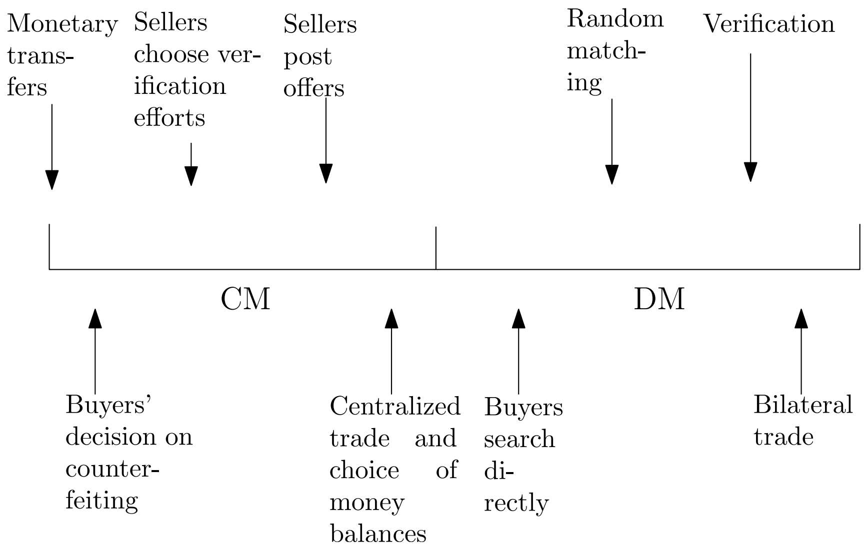

Figure 3: Timing of Events in a period

The timing of events in a period is summarized in Figure 3.[9] At the beginning of each period, money is transferred from or back to the government. After that, buyers decide whether to produce counterfeits or not and if so, what quality of counterfeits to produce. This decision is private information. Sellers decide whether to verify or not the money used by the buyer by investing in the verification technology. This decision may or may not be observable and we will consider both cases below. After that, sellers post their offers for the night market. Then the Walrasian market opens in which agents consume and produce as well as adjust their money balances. When night falls, the Walrasian market closes, and the competitive search market opens. Sub-markets are formed as a result of the competitive search process. When a buyer and a seller meet in a sub-market, the buyer hands in his holdings of money to the seller. If the seller has invested in verification, she will receive a signal. She will become informed or not depending on the signal that she receives. The pair trades according to the pre-specified terms of trade: the payment is made; the seller produces; and the buyer consumes. After matches are terminated, all agents learn the type of their money holdings and all counterfeits disintegrate.

Remark 1. In the bilateral trading, once the buyer has surrendered his money holdings to the seller for verification, he does not have an incentive to take them back and replace any genuine money with counterfeits, even if he now learns that the seller cannot verify. It is because by doing so, he is signaling to the seller that the bank notes he is now giving the sellers are fake. Clearly no trade will occur since sellers never accept counterfeits knowingly. Therefore, once buyers have accepted the offer,

[9] Here we consider the case where the probability of the signal being informative depends on the actions of the buyers and sellers. For the baseline case where the probability is exogenous, there will not be decisions on counterfeit quality and verification.

they will leave whatever money that they held on the table. This immediately implies that if a buyer decides to produce counterfeits with quality $h > 0$, he will not spend any genuine notes in the search market, i.e., genuine notes and counterfeits are not combined together as a payment. The reason is that combining genuine and counterfeit notes does not increase the chance that counterfeits can pass through as the probability of detection depends on the quality but not the quantity of counterfeits. Since the buyer has already incurred the sunk cost of producing counterfeits, the above strategy is strictly dominated by holding the same amount of genuine notes but producing no counterfeits (or counterfeits with quality $h = 0$). Hence, we can further restrict the strategy space for buyers such that they will not hold a portfolio of genuine and counterfeit notes. In other words, if a buyer produces counterfeit notes, he will use only counterfeits to buy goods in the search market.

3 Counterfeiting in Competitive Search

Here we consider a counterfeiting setup similar to Nosal and Wallace (2007), except that we consider competitive search as the trading mechanism instead of the buyer making a take-it-or-leave it offer. We abstract from the counterfeiter's decision regarding quality of counterfeits. Thus a buyer can produce counterfeit notes in any quantity at a fixed cost $g > 0$. We also abstract from the seller's verification decision and assume that the seller will always receive a signal about the quality of the buyer's notes. With probability π, the seller knows whether the money is genuine or counterfeit, while with probability $1 - \pi$, the seller is uninformed. Since counterfeit notes are 100% confiscated by the government at the end of each period, Lemma 1 states that the only possible offer is pooling.

Lemma 1. . *When sellers are informed, there is no trade between a seller and a counterfeiter, i.e.* $q_1^c = 0$. *When sellers are uninformed, there is no separating offer, i.e.* $q_0^g = q_0^c = q_0$.

Lemma 1 contrasts our model with Guerrieri et al. (2010) in the sense that the equilibrium in our model involves pooling. The reason is that the sorting assumption in Guerrieri et al. (2010) does not hold here.[10]

Let P_t be the fraction of genuine money holders in the economy and $W_t^b(m_t)$ be the value function of a buyer who enters the Walrasian market at time t holding m_t units of money.[11] Since the buyer needs to choose whether to produce a counterfeit

[10]Note that the trade surplus in a match (given the buyer's decision on counterfeiting) when the seller is uninformed is: $u(q_0) - \beta\phi d_0/\gamma$ for the genuine money holder, and $u(q_0)$ for the counterfeiter. There is no way for the seller to post an offer (q_0, d_0) to screen buyers since a counterfeiter can always duplicate the strategy of a buyer holding genuine money.

[11]To simplify the notation, we suppress all aggregate state variables but a money holding into a subscript t for all the value functions through out the paper.

or not, his value function at the start of each period is

$$W_t^b(m_t) = \max_{p_t} \left[p_t W_t^g(m_t) + (1-p_t) W_t^c(m_t)\right], \tag{3}$$

where W^g and W^c represent the value functions of being a holder of genuine money or a counterfeiter respectively, and p is the choice on counterfeits (or holding genuine money). In equilibrium $p = P$. The optimal decision on P must satisfy

$$\begin{cases} P = 1 & \text{if } W^g > W^c, \\ P = 0 & \text{if } W^g < W^c, \\ P \in (0,1) & \text{if } W^g = W^c. \end{cases} \tag{4}$$

Denote the buyer's value function of carrying $\hat{m}_t$ dollars into the search market of period t by $V_t^g(\hat{m}_t)$ if he uses only genuine notes, or $V_t^c(\hat{m}_t)$ otherwise. Similarly, $W_t^s(m_t)$ and $V_t^s(\hat{m}_t)$ are the corresponding value functions for the seller. Let the price of the consumption good in the Walrasian market be normalized to 1 and denote the price of a dollar in units of consumption in period t by ϕ_t. The value of the agent $j \in \{s, g, c\}$ in the Walrasian market is

$$W_t^j(m_t) = \max_{x_t, y_t, \hat{m}_t} \left[v(x_t) - y_t + V_t^j(\hat{m}_t)\right], \tag{5}$$

$$s.t.\ x_t + \phi_t \hat{m}_t = y_t + \phi_t(m_t + \tau_t), \tag{6}$$

where τ_t is the nominal monetary transfers to (or from) the agent.[12]

Because utility is quasi-linear, the budget constraint (6) can be substituted into the objective function (5), so that the problem simplifies to

$$W_t^j(m_t) = \max_{x_t, \hat{m}_t} \left[v(x_t) - x_t - \phi_t(\hat{m}_t - m_t - \tau_t) + V_t^j(\hat{m}_t)\right]. \tag{7}$$

Thus, it follows that for $j \in \{s, g, c\}$

1. the optimal choice of x_t is independent of m_t with $v'(x_t) = 1$, so $x_t = x^*$;

2. the optimal choice of $\hat{m}_t$ is also independent of m_t, and is determined by maximizing $V_t^j(\hat{m}_t) - \phi_t \hat{m}_t$;

3. the value functions $W_t^j(m_t)$ are linear in m_t and can be rewritten as

$$W_t^j(m_t) = W_t^j(0) + \phi_t m_t.$$

The value functions of buyers and sellers at night depend on the submarket they visit in equilibrium, and on the money holdings they take into this sub-market. All agents have rational expectations regarding the number of buyers will be attracted by

[12] The choices of x_t, y_t and $\hat{m}_t$ are conditional on j, but we omit the superscript to ease notation.

each offer that sellers post, and thus about the market tightness in each sub-market. Sellers have beliefs on the fraction of genuine money holders in each sub-market. The set of offers posted in equilibrium must be such that sellers have no incentives to post deviating offers. Therefore, a sub-market is characterized by the fraction of genuine money holders P, a market tightness θ, and an offer $\{q_\varepsilon^i, d_\varepsilon^i\}$. Let Ω be the set of all sub-markets that are active in equilibrium. An element $\omega \in \Omega$ is then a list $\omega = \{P, \theta, \{q_\varepsilon^i, d_\varepsilon^i\}\}$. Competition between sellers on posting will force the value of all active sub-markets to be equal.

Given Lemma 1 and that all the sellers are identical, it is obvious that there is only one active sub-market at night. Thus we can write down the value function for the sellers at night as

$$\begin{aligned} V_t^s(\hat{m}_t) &= \alpha^s(\theta_t) P_t \left[\begin{array}{l} \pi\left(-q_{1t} + \beta W_{t+1}^s(\hat{m}_t + d_{1t})\right) \\ +(1-\pi)\left(-q_{0t} + \beta W_{t+1}^s(\hat{m}_t + d_{0t})\right) \end{array} \right], \\ &\quad + \alpha^s(\theta_t)(1-P_t)\left[(1-\pi)(-q_{0t}) + \beta W_{t+1}^s(\hat{m}_t)\right] \\ &\quad + (1-\alpha^s(\theta_t))\beta W_{t+1}^s(\hat{m}_t) \\ &= \alpha^s(\theta_t) P_t \left[\begin{array}{l} \pi(-q_{1t} + \beta\phi_{t+1} d_{1t}) \\ +(1-\pi)(-q_{0t} + \beta\phi_{t+1} d_{0t}) \end{array} \right]. \\ &\quad + \alpha^s(\theta_t)(1-P_t)\left[(1-\pi)(-q_{0t})\right] \\ &\quad + \beta\left[\phi_{t+1}\hat{m}_t + W_{t+1}^s(0)\right] \end{aligned} \tag{8}$$

With probability $\alpha^s(\theta_t)$, a seller will meet with a potential buyer. Conditional on a successful match, the seller will meet a buyer with genuine money with probability P_t. With probability $1 - P_t$, the seller will meet with a counterfeiter and will suffer a loss in the case when he is uninformed. Notice that to derive (8), we use the fact that $W_{t+1}^s(\hat{m}_t)$ is linear with slope ϕ_{t+1}. Combining (7) and (8), the optimal choice of $\hat{m}_t$ solves

$$\max_{\hat{m}_t \geq 0} (\beta\phi_{t+1} - \phi_t)\hat{m}_t.$$

The solution exists if and only if the inflation rate $\phi_t/\phi_{t+1} > \beta$. In this case $\hat{m}_t = 0$ and the seller will not carry money to the search market because he cannot derive any benefit from holding money.

Similarly, the value functions for a buyer at night are, if he is holding genuine money,

$$\begin{aligned}
V_t^g\left(\hat{m}_t\right) &= \alpha^b\left(\theta_t\right)\left[\begin{array}{l}\pi\left(u\left(q_{1t}\right)+\beta W_{t+1}^b\left(\hat{m}_t-d_{1t}\right)\right) \\ +\left(1-\pi\right)\left(u\left(q_{0t}\right)+\beta W_{t+1}^b\left(\hat{m}_t-d_{0t}\right)\right)\end{array}\right] \\
&\quad +\left(1-\alpha^b\left(\theta_t\right)\right)\beta W_{t+1}^b\left(\hat{m}_t\right) \\
&= \alpha^b\left(\theta_t\right)\left[\begin{array}{l}\pi\left(u\left(q_{1t}\right)-\beta\phi_{t+1}d_{1t}\right) \\ +\left(1-\pi\right)\left(u\left(q_{0t}\right)-\beta\phi_{t+1}d_{0t}\right)\end{array}\right] \qquad (9) \\
&\quad +\beta\left[\phi_{t+1}\hat{m}_t+W_{t+1}^b\left(0\right)\right] \\
s.t.\; d_{1t} &\leq \hat{m}_t, \qquad (10) \\
d_{0t} &\leq \hat{m}_t. \qquad (11)
\end{aligned}$$

And if he is a counterfeiter,

$$\begin{aligned}
V_t^c\left(\hat{m}_t\right) &= \max_{h_t} -g+\alpha^b\left(\theta_t\right)\left(1-\pi\right)u\left(q_{0t}\right)+\beta W_{t+1}^b\left(\hat{m}_t\right) \\
&= \max_{h_t} -g+\alpha^b\left(\theta_t\right)\left(1-\pi\right)u\left(q_{0t}\right)+\beta\left[\phi_{t+1}\hat{m}_t+W_{t+1}^b\left(0\right)\right]. \qquad (12)
\end{aligned}$$

Here, the counterfeiter can extract information rent from a seller who is uninformed about the quality of his notes. Again, combining (7) and (12) shows that the counterfeiter carries no genuine money to the search market.

Plug the value functions at night into the value function at the beginning of the day, the daytime values for buyers and sellers can be rewritten as follows:

$$\begin{aligned}
W_t^b\left(m_t\right) &= v\left(x^*\right)-x^*+\phi_t\left(m_t+\tau_t\right)+\beta W_{t+1}^b\left(0\right)+\max_{p_t}\left[p_t S_t^g+\left(1-p_t\right)S_t^c\right], \\
W_t^s\left(m_t\right) &= v\left(x^*\right)-x^*+\phi_t\left(m_t+\tau_t\right)+\beta W_{t+1}^b\left(0\right).
\end{aligned}$$

Next we can write down the expected trade surpluses of an agent at night as

$$\begin{aligned}
S_t^g &= \alpha^b\left(\theta_t\right)\left[\begin{array}{l}\pi\left(u\left(q_{1t}\right)-\beta\phi_{t+1}d_{1t}\right) \\ +\left(1-\pi\right)\left(u\left(q_{0t}\right)-\beta\phi_{t+1}d_{0t}\right)\end{array}\right]-\left(\phi_t-\beta\phi_{t+1}\right)\hat{m}_t, \\
S_t^c &= \alpha^b\left(\theta_t\right)\left(1-\pi\right)u\left(q_{0t}\right)-g, \\
S_t^s &= \alpha^s\left(\theta_t\right)\left\{\begin{array}{l}P_t\left[\begin{array}{l}\pi\left(-q_{1t}+\beta\phi_{t+1}d_{1t}\right) \\ +\left(1-\pi\right)\left(-q_{0t}+\beta\phi_{t+1}d_{0t}\right)\end{array}\right] \\ -\left(1-P_t\right)\left(1-\pi\right)q_{0t}\end{array}\right\}.
\end{aligned}$$

In terms of the surpluses, condition (4) on P can be rewritten as

$$\begin{cases} P=1 & \text{if } S^g>S^c, \\ P=0 & \text{if } S^g<S^c, \\ P\in(0,1) & \text{if } S^g=S^c. \end{cases} \qquad (13)$$

We confine our attention to symmetric steady state equilibrium where the real money balance remains constant. Thus, $\phi_{t+1}M_{t+1} = \phi_t M_t$ implies that the inflation rate equals the money growth rate: $\phi_t/\phi_{t+1} = \gamma$. Let $\overline{S}^j$ be the equilibrium expected surplus at night for $j \in \{s, g, c\}$. We define the competitive search equilibrium as follows.

Definition 1. *A competitive search equilibrium consists of value functions W^b and W^s, a set $\left\{\Omega, \overline{S}^g, \overline{S}^c, \overline{S}^s\right\}$, an aggregate state P, a price ϕ, and the buyer's decision on p, such that, for all $\omega \in \Omega$,*

1. *Symmetry: $p = P$.*
2. *The buyers' optimal choice of P must satisfy* (13)*.*
3. *All genuine money holders attain the same expected surplus $\overline{S}^g \geq 0$.*
4. *All counterfeiters attain the same expected surplus $\overline{S}^c \geq 0$.*
5. *Free entry implies sellers' expected surplus equal to 0.*
6. *The list ω solves the following program:*

$$\overline{S}^s = \max_{\theta,q_0,q_1,d_0,d_1,\hat{m}} \left\{ \begin{array}{l} \alpha^b(\theta) \left[\begin{array}{l} \pi \left(u(q_1) - \beta \dfrac{\phi}{\gamma} d_1 \right) \\ + (1-\pi) \left(u(q_0) - \beta \dfrac{\phi}{\gamma} d_0 \right) \end{array} \right] \\ - \dfrac{\gamma - \beta}{\gamma} \hat{m} \phi_t \end{array} \right\}, \quad (14)$$

$$\text{s.t. } d_0 \leq \hat{m}, \quad (15)$$

$$d_1 \leq \hat{m}, \quad (16)$$

$$\overline{S}^c = \alpha^b(\theta)(1-\pi) u(q_0) - g, \quad (17)$$

$$\overline{S}^y = \alpha^s(\theta) \left\{ \begin{array}{l} P \left[\begin{array}{l} \pi \left(\beta \dfrac{\phi}{\gamma} d_1 - q_1 \right) \\ + (1-\pi) \left(\beta \dfrac{\phi}{\gamma} d_0 - q_0 \right) \end{array} \right] \\ - (1-P)(1-\pi) q_0 \end{array} \right\} = 0. \quad (18)$$

Conditions 1 to 5 are straightforward. Since the same type of buyers have identical payoff functions, they must attain the same expected surplus. Given that sellers are free to enter any one of the active submarkets, in equilibrium the expected surplus of a seller must be zero. Condition 6 results from a combination of optimal behavior and competition among sellers when they post offers. According to this condition, sellers cannot post offers that attract more genuine money holders without making himself worse off. Since there are infinitely many counterfeiters entering the submarket as well, each sellers takes $\overline{S}^c$ as given. Sellers also realize that the ratio θ is going to adjust endogenously so that (17) and (18) hold.

To characterize the equilibrium, we first state the following lemma to describe two properties of the program in (14) to (18).

Lemma 2. *The optimal payments are uniform and satisfy:* $d_0 = d_1 = \hat{m}$.

The intuition is rather simple. Since carrying genuine money is costly owing to inflation, buyers will carry just enough money when entering the search market to cover their largest possible payment. If the two payments are different, then there exists a linear combination of the two payments which satisfies all the constraints. However, a buyer can now carry less money under the new payments and thus is better off. Therefore the two payments cannot be different.

Lemma 3. *Buyers and sellers trade with probability one in any active submarket:* $\theta = \alpha^b(\theta) = \alpha^s(\theta) = 1$.

This lemma is a direct corollary of the assumption regarding the matching process. Note also that there is only one active submarket. If $\theta > 1$, for example, then not all buyers will be matched while all sellers will be matched for sure. Consider a decrease in θ, due to say entry of additional sellers. Then buyers will now have a higher matching probability. It is possible to find another pair of quantities that satisfy all constraints while offering a higher quantity of goods when genuine money is used and a lower quantity if sellers are uninformed. Obviously, buyers who use genuine money are better off in this case as they have a higher probability of matching and a higher quantity of goods traded. Therefore, $\theta > 1$ cannot be sustained. Similar arguments apply to the case of $\theta < 1$.

We can further simplify the program using Lemma 2 and 3 to show the first proposition regarding the property of posted prices in equilibrium.

Proposition 1. *In any monetary equilibrium with counterfeiting,* $q_0 \leq q^{**} < q$ *for any* $P \in (0,1)$*, where* $q^{**} = \arg\max_q u(q) - \gamma/(\beta P)\, q$.

The intuition behind Proposition 1 is as follows. By Lemma 2, a seller will charge the same amount of payment whether he is informed or not. However, if the seller cannot recognize the quality of the buyer's money holding, he must produce less than (or post higher price than in) the case when he is informed, in order to compensate for the risk that he may receive counterfeit notes.

Proposition 1 is a necessary condition for the existence of monetary equilibrium with counterfeiting. It implies that the set of submarkets is *complete* in the sense that there is no profitable deviation for sellers to open another submarket which only attracts genuine money holders. The condition in Proposition 1 rules out any incentives for sellers to deviate from the pooling equilibrium. A seller will deviate from a pooling contract only if he can attract buyers using genuine money but not counterfeiters. Thus the only potential deviation is to offer a lower q_0 and d_0 in an attempt to attract buyers with genuine money since counterfeiters care only about the quantity of goods traded. However, given Proposition 1 and the concavity of $u(q)$, any such offers will make the buyer using genuine money straightly worse off if the seller wants to make at least zero profits.

Since the counterfeiter's surplus must be non-negative, the immediate corollary implies that $g < (1-\pi)u(q^{**})$ in counterfeiting equilibrium. The next proposition characterize a sufficient condition for the existence of monetary equilibrium with counterfeiting.

Proposition 2. *A monetary equilibrium with counterfeiting exists if*

$$g < \frac{\gamma}{\beta}q_1 - \pi u(q_1). \tag{19}$$

where q_1 satisfies $u'(q_1) = \gamma/\beta$.

According to Proposition 2, as long as the cost of producing counterfeits is sufficiently low, a monetary equilibrium with counterfeiting will exist. This result is in contrast to the one found in Nosal and Wallace (2007) where counterfeiting is merely a threat to the monetary economy: if the cost of producing counterfeits is low, then no trade will take place, while if the cost is high, no one will produce counterfeits. What is driving our results is the different pricing mechanism used in our study. The non-existence of counterfeiting in Nosal and Wallace is due to the fact that a genuine money holder can always signal her type through the offer she makes. While in our case, sellers post offers so that buyers cannot signal the quality of their money holding via their offers. More importantly, if sellers are uninformed about the quality of the money used by buyers, the offers must be pooling since counterfeits have no value. Therefore counterfeiters can extract rents and sellers will accept counterfeits only when they cannot recognize the quality of the money used.[13]

4 The seller's verification decision is known

In the previous section, we have shown that counterfeiting can exist as an equilibrium outcome under competitive search. In the next two sections, we consider the case where the probability of the signal about the quality of the buyer's money being

[13] Although our result involves pooling offers, a monetary equilibrium can still exist. This is not the case in Guerrieri et al. (2010) where they show that a pooling offer will cause the asset market to shut down. The main reason why trades can happen in our model but not in their example is that the principal (seller) in our model always has some positive probability of knowing the agent's (buyer) type which guarantees trades to take place. While in Guerrieri et al. (2010), the principal has no information about the agent's type at all, and thus the absence of trade surplus results in no trade.

informative is endogenous. That is, the probability depends on the quality of the counterfeits and the seller's decision on verification. In this case, we can study how public policies will affect counterfeiting. In general, a seller's verification decision may or may not be known to others. In this section, we look at the case where this decision is public information and we will consider the other case in the next section.

Again, there is no separating offer when sellers are uninformed. Since sellers can choose to verify the money they receive or not and such a decision is known to everybody. Thus, buyers know the type of sellers that they will meet and sellers have beliefs on the fraction of genuine money holders in each sub-market. The set of offers posted in equilibrium must be such that sellers have no incentives to post deviating offers. Therefore, a sub-market is characterized by seller's type a, the fraction of genuine money holders P_a, a market tightness θ_a, and an offer $\{q_\varepsilon^i, d_\varepsilon^i\}_a$. Let Ω be the set of all sub-markets that are active in equilibrium. An element $\omega \in \Omega$ is then a list $\omega = \{a, P_a, \theta_a, \{q_\varepsilon^i, d_\varepsilon^i\}_a\}$. Competition between sellers on posting will force the value of all active sub-markets to be equal. The following lemma shows that the only active sub-market at night is where the sellers choose to invest in the verification technology: $a = y$.

Lemma 4. *The only active sub-market in equilibrium is the one where $a = y$.*

Note that Lemmas 2 and 3 will continue to hold. Given these lemmas, we can write down the value function for the sellers at night as

$$V_t^s(\hat{m}_t) = \max\{V_t^y(\hat{m}_t), V_t^n(\hat{m}_t)\}$$

where $V_t^n(\hat{m}_t) = \beta W_{t+1}^s(\hat{m}_t) = \beta\left[\phi_{t+1}\hat{m}_t + W_{t+1}^s(0)\right]$, and

$$\begin{aligned}
V_t^y(\hat{m}_t) \quad = \quad & -L + \alpha^s(\theta_t) P_t \begin{bmatrix} \pi(H_t)\left(-q_{1t} + \beta W_{t+1}^s(\hat{m}_t + d_{1t})\right) \\ +(1-\pi(H_t))\left(-q_{0t} + \beta W_{t+1}^s(\hat{m}_t + d_{0t})\right) \end{bmatrix}, \\
& + \alpha^s(\theta_t)(1-P_t)\left[(1-\pi(h_t))(-q_{0t}) + \beta W_{t+1}^s(\hat{m}_t)\right] \\
& + (1-\alpha^s(\theta_t))\beta W_{t+1}^s(\hat{m}_t) \\
= \quad & -L + \alpha^s(\theta_t) P_t \begin{bmatrix} \pi(H_t)(-q_{1t} + \beta\phi_{t+1}d_{1t}) \\ +(1-\pi(H_t))(-q_{0t} + \beta\phi_{t+1}d_{0t}) \end{bmatrix}. \\
& + \alpha^s(\theta_t)(1-P_t)\left[(1-\pi(h_t))(-q_{0t})\right] \\
& + \beta\left[\phi_{t+1}\hat{m}_t + W_{t+1}^s(0)\right]
\end{aligned} \tag{20}$$

If a seller chooses not to verify, she will attract only the buyers who have produced counterfeit notes of the lowest quality. Not willing to accept counterfeits since they will completely disintegrate at the end of the period, the seller will not trade at all. When a seller exerts an effort in verification, he will have to pay a cost L. With probability $\alpha^s(\theta_t)$, he will meet with a potential buyer. Conditional on a successful match, the seller will produce for genuine money with probability P_t. With probability

$1-P_t$, the seller will meet with a counterfeiter and will suffer a loss in the case when he is uninformed.

Similarly, the value functions for a buyer at night are, if he is holding genuine money,

$$
\begin{aligned}
V_t^g(\hat{m}_t) &= \alpha^b(\theta_t)\begin{bmatrix} \pi(H_t)\left(u(q_{1t})+\beta W_{t+1}^b(\hat{m}_t-d_{1t})\right) \\ +(1-\pi(H_t))\left(u(q_{0t})+\beta W_{t+1}^b(\hat{m}_t-d_{0t})\right)\end{bmatrix} \\
&\quad +\left(1-\alpha^b(\theta_t)\right)\beta W_{t+1}^b(\hat{m}_t) \\
&= \alpha^b(\theta_t)\begin{bmatrix} \pi(H_t)(u(q_{1t})-\beta\phi_{t+1}d_{1t}) \\ +(1-\pi(H_t))(u(q_{0t})-\beta\phi_{t+1}d_{0t})\end{bmatrix} \qquad (21) \\
&\quad +\beta\left[\phi_{t+1}\hat{m}_t+W_{t+1}^b(0)\right] \\
s.t.\ d_{1t} &\leq \hat{m}_t, \qquad (22) \\
d_{0t} &\leq \hat{m}_t. \qquad (23)
\end{aligned}
$$

and if he is a counterfeiter,

$$
\begin{aligned}
V_t^c(\hat{m}_t) &= \max_{h_t} -g(h_t)+\alpha^b(\theta_t)(1-\pi(h_t))u(q_{0t})+\beta W_{t+1}^b(\hat{m}_t) \\
&= \max_{h_t} -g(h_t)+\alpha^b(\theta_t)(1-\pi(h_t))u(q_{0t})+\beta\left[\phi_{t+1}\hat{m}_t+W_{t+1}^b(0)\right] \qquad (24)
\end{aligned}
$$

It can be shown easily that the counterfeiters and sellers carry no genuine money to the search market. The optimal choice of h is given by the following first order condition, which equates the marginal cost to the marginal benefit of increasing counterfeit quality:

$$
g'(h_t)=-\pi'(h_t)u(q_{0t})\,\alpha^b(\theta_t). \qquad (25)
$$

Plug the value functions at night into the value function at the beginning of the day, the daytime values for buyers and sellers can be rewritten as follows:

$$
\begin{aligned}
W_t^b(m_t) &= v(x^*)-x^*+\phi_t(m_t+\tau_t)+\beta W_{t+1}^b(0)+\max_{p_t}\left[p_t S_t^g+(1-p_t)S_t^c\right], \\
W_t^s(m_t) &= v(x^*)-x^*+\phi_t(m_t+\tau_t)+\beta W_{t+1}^b(0)+\max\left\{0,S_t^y\right\}.
\end{aligned}
$$

Here S^j, $j\in\{g,c,y\}$, represents the expected trade surpluses of an agent at night which can be written as

$$
\begin{aligned}
S_t^g &= \alpha^b(\theta_t)\begin{bmatrix} \pi(H_t)(u(q_{1t})-\beta\phi_{t+1}d_{1t}) \\ +(1-\pi(H_t))(u(q_{0t})-\beta\phi_{t+1}d_{0t})\end{bmatrix}-(\phi_t-\beta\phi_{t+1})\hat{m}_t, \\
S_t^c &= \alpha^b(\theta_t)(1-\pi(h_t))u(q_{0t})-g(h_t), \\
S_t^y &= \alpha^s(\theta_t)\left\{\begin{array}{l} P_t\begin{bmatrix} \pi(H_t)(-q_{1t}+\beta\phi_{t+1}d_{1t}) \\ +(1-\pi(H_t))(-q_{0t}+\beta\phi_{t+1}d_{0t})\end{bmatrix} \\ -(1-P_t)(1-\pi(h_t))q_{0t}\end{array}\right\}-L.
\end{aligned}
$$

We define the competitive search equilibrium in this case as follows.

Definition 2. *A competitive search equilibrium consists of value functions W^b and W^s, a set $\left\{\Omega, \overline{S}^g, \overline{S}^c, \overline{S}^y\right\}$, aggregate states P and H, a price ϕ, and individuals decisions on p and h, such that, for all $\omega \in \Omega$,*

1. *Symmetry: $p = P$ and $h = H$.*
2. *The buyer's optimal choice of P must satisfy* (13).
3. *The quality of counterfeits should satisfy* (25).
4. *All genuine money holders attain the same expected surplus $\overline{S}^g \geq 0$.*
5. *All counterfeiters attain the same expected surplus $\overline{S}^c \geq 0$.*
6. *Free entry implies seller's expected surplus equal to 0.*
7. *The list ω solves the following program:*

$$\begin{aligned}
\overline{S}^g &= \max_{\theta, q_0, q_1, d_0, d_1, \hat{m}} \left\{ \begin{array}{l} \alpha^b(\theta) \left[\begin{array}{l} \pi(H)\left(u(q_1) - \beta \dfrac{\phi}{\gamma} d_1\right) \\ + (1-\pi(H))\left(u(q_0) - \beta \dfrac{\phi}{\gamma} d_0\right) \end{array} \right] \\ - \dfrac{\gamma - \beta}{\gamma} \hat{m} \phi_t \end{array} \right\}, & (26) \\
s.t.\; d_0 &\leq \hat{m}, & (27) \\
d_1 &\leq \hat{m}, & (28) \\
\overline{S}^c &= \alpha^b(\theta)(1-\pi(H))\, u(q_0) - g(H), & (29) \\
\overline{S}^y &= \alpha^s(\theta) \left\{ \begin{array}{l} P \left[\begin{array}{l} \pi(H)\left(\beta \dfrac{\phi}{\gamma} d_1 - q_1\right) \\ + (1-\pi(H))\left(\beta \dfrac{\phi}{\gamma} d_0 - q_0\right) \end{array} \right] \\ - (1-P)(1-\pi(H))\, q_0 \end{array} \right\} - L \\
&= 0. & (30)
\end{aligned}$$

Since the counterfeiters can choose a quality of counterfeits to produce, we have the additional condition 3 regarding the quality of counterfeits. We also restate Proposition 1 below as Proposition 3 since the proof is slightly different in this case.

Proposition 3. *In any monetary equilibrium with counterfeiting, $q_0 \leq q^{**} < q$ for any $P \in (0,1)$, where $q^{**} = \arg\max_q u(q) - \gamma / (\beta P)\, q$.*

Next we are going to characterize the conditions under which a monetary equilibrium exists.

Proposition 4. *A monetary equilibrium exists if and only if* $(\gamma, L) \in A$*, where*

$$A \equiv \left\{ (\gamma, L) \in [\beta, +\infty) \times (0, +\infty) \middle| \pi(0) \left(u(q_1) - \frac{\gamma}{\beta} q_1 \right) \geq \frac{\gamma}{\beta} L \right\}, \tag{31}$$

and q_1 *satisfies*

$$u'(q_1) = \frac{\gamma}{\beta}. \tag{32}$$

Moreover, a monetary equilibrium with counterfeiting exists if and only if $(\gamma, L) \in int(A)$.

Essentially, Proposition 4 says that if the inflation rate and the verification cost are not too high, there exists a monetary equilibrium in which both genuine and counterfeit money coexist. If the verification cost is too high, for example, sellers are not willing to invest in verification and thus remain uninformed about the quality of notes received from buyers. In this case, there will be no monetary equilibrium.

Result 1. *Suppose* $(\gamma, L) \in int(A)$*, for any given* L*, there is a monetary equilibrium with counterfeiting such that*

1. $dP/d\gamma > 0$ for given q_0;
2. $dq_0/d\gamma < 0$ for given P.

Result 1 says that a higher money growth rate and thus inflation will increase the share of buyers using genuine money and lower the quantity traded when the seller is uninformed. Thus higher inflation tends to reduce counterfeiting.

Result 2. *Suppose* $(\gamma, L) \in int(A)$*, for any given* γ*, there is a monetary equilibrium with counterfeiting such that*

1. $dP/dL > 0$ for given q_0;
2. $dq_0/dL < 0$ for given P.

Result 2 says that a higher cost of verification for the seller leads to a higher share of buyers using genuine money and a lower quantity traded when the seller is uninformed. The latter result is rather straightforward. For a given fraction of buyers using genuine money, a seller must reduce the quantity traded when uninformed in order to offset the higher cost of verification. The former result, however, appears to be counter-intuitive as one would expect a higher cost of verification will result in fewer verification and thus more counterfeiting. Since counterfeit notes are worthless to the sellers, they can gain from trade only if the buyer uses genuine money. If a seller does not verify, only counterfeiters will be attracted to the submarket. Therefore, a higher cost of verification implies that there must be more buyers using genuine money in order to entice sellers to participate in the decentralized market.

Finally, we look at the policy that makes bank notes more difficult to counterfeit. To do that, assume the cost function takes the form $g(h) = \delta g_0(h)$ where $g_0' > 0$, $g_0'' > 0$ and $\delta > 1$ is the policy parameter that represents the security features of bank notes. That is, a higher δ implies more security features of a bank note and thus it is more difficult to counterfeit. The following proposition shows the two sided effects of such a policy.

Result 3. *For any given $(\gamma, L) \in int(A)$, the effect of a change in δ is*

1. $dP/d\delta > 0$ for given H;
2. $dP/d\delta < 0$ for given q_0.

According to Result 3, increasing the cost of counterfeiting, by for example adding more security features to a bank note, does not necessarily reduce counterfeiting. For a given quality of counterfeits, an increase in the cost of producing the given quality while not changing the probability of detection, will make producing counterfeit notes less profitable. As a result, there will be less buyers producing counterfeit notes and thus the fraction of buyers using geniune money increases. If q_0 is given, in response to an increase in the cost of producing counterfeits, buyers using counterfeit notes will lower the quality of counterfeits. Given that buyers can still get the same q_0, a lower quality of counterfeits results in more counterfeiting and thus a lower fraction of buyers using genuine money. Thus a higher cost of counterfeiting for given q_0 leads to more counterfeiting.

The implication of Results1 to 3 is that a single anti-counterfeiting measure may not be able to reduce counterfeiting. It is thus more effective by combining different anti-counterfeiting measures such as adding more security features and lowering the cost of verification.

5 The seller's verification decision is private information

Now we consider the case when the seller's decision on verification is not public information. We will show that the basic results in the previous section remain unchanged. In addition, it is now possible to fully pin down the equilibrium outcome.

In order for the seller to truthfully reveal whether he is informed or not, there should be no gain by pretending to be uninformed when indeed he is informed and vice versa. Thus we have the following incentive compatibility constraint:

$$\pi(q_0 - Pq_1) = L. \tag{33}$$

Proposition 5. *In any pooling monetary equilibrium with counterfeiting, $Pq_1 \leq q_0 \leq q_1$ for any $P \in (0,1)$.*

A pooling monetary equilibrium exists if and only if $(\gamma, L) \in B$, *where*

$$B \equiv \left\{ (\gamma, L) \in [\beta, +\infty) \times (0, +\infty) \left| \begin{array}{c} \pi u(q_1) - \frac{\gamma}{\beta} q_1 + g \geq \frac{\gamma}{\beta} \frac{L}{\pi} \\ u\left(\frac{L}{\pi} + q_1\right) \pi' + g' = 0 \end{array} \right. \right\}, \tag{34}$$

and q_1 *satisfies* (32). *Moreover, monetary equilibrium with counterfeiting exists if and only if* $(\gamma, L) \in int(B)$.

Again, proposition 5 says that if the inflation rate and the verification cost are not too high, there exists a monetary equilibrium in which both genuine and counterfeit money coexist. Note that the additional equation 33 allows us to determine all the variables in the model.

Assumption 1. *For any* $q \in (0, q^*]$, $-qu''/u' > 1-\pi^*$, *where* $\pi^* = \pi(H(q^*))$ *satisfies* $u(q^*)\pi'(H) + g'(H) = 0$.

Next we examine the effects of changes in the inflation rate, cost of verification and cost of producing counterfeit notes on the fraction of buyers using genuine money and the quality of counterfeit notes produced.

Result 4. *Under assumption 1, the pooling equilibrium with counterfeiting has the following properties:*

1. $dP/d\gamma > 0$, and $dH/d\gamma < 0$;
   (a) $dP/dL > 0$, and $dH/dL < 0$;
   (b) $dP/d\delta < 0$, and $dH/d\delta > 0$.

Note that these results are fairly similar to those in Results 1 to 3. One major difference, however, is that these results now hold in general rather than for any given values of some of the variables.

6 Conclusion

In this paper, we have constructed a monetary search model in which money is not perfectly recognizable and thus can be counterfeited at a cost. We show that, under certain fairly general conditions, a monetary equilibrium with counterfeiting will exist. This result is more consistent with the observation that in recent years some countries, such as Canada and the United Kingdom, have experienced high levels of counterfeiting of bank notes. Our model differs from other existing search model of counterfeiting by using a competitive search environment rather than the typical random search setup. In such an environment, buyers cannot signal to sellers whether they are using genuine money or not. Thus the resulting pooling equilibrium allows the use of both genuine and counterfeit notes. We also model explicitly the interaction between the seller's verification decision and the buyer's choice of quality of counterfeits. We find that a combination of anti-counterfeiting measures is more effective in reducing counterfeiting than any single measure.

It would be of interest to relax some of the assumptions in our model; for example, counterfeit notes cannot circulate across periods because they disintegrate or are confiscated completely at the end of the period. Under this assumption, sellers will accept counterfeits only unknowingly because counterfeits have no value. In practice, however, counterfeit bank notes tend to circulate for a short while before they are detected and sent to law enforcement agencies. This is especially true since most central banks do not process the notes in circulation every period and some central banks also delegate note processing to financial institutions. In this case, sellers may accept counterfeit notes knowingly because counterfeit notes could have value in exchanges if other agents are willing to accept them as well. In future extensions, it would be of interest to consider the case that the central bank determines the rate of confiscation of counterfeits and how that decision affects counterfeiting.

A Appendix

A.1 Proof of Lemma 1

Proof. Sellers do not produce in exchange for a counterfeit because by assumption, counterfeits have no future values. If there exists a separating contract when sellers are uninformed, it implies that $q_0^g \neq q_0^c$. If $q_0^g > q_0^c$, then a counterfeiter will always prefer the contract for genuine money holders since a counterfeiter can produce any amount of counterfeited money of the same quality at the same cost. If $q_0^g < q_0^c$, then a seller can offer q_0', where $q_0^g < q_0' < q_0^c$, and attract all the buyers holding genuine money. Therefore, when sellers are uninformed, there will not be a separate offer. $\square$

A.2 Proof of Lemma 2

Proof. First notice that either one of the liquidity constraints in (15) and (16) must bind and $\hat{m} = \max\{d_0, d_1\}$. Then by contradiction, suppose $d_0 \neq d_1$ for any offer $\{(q_0, d_0), (q_1, d_1)\}$ in equilibrium. Take $\hat{d} = \pi d_1 + (1-\pi) d_0 < \max\{d_0, d_1\} = \hat{m}$. Consider another offer where the same amount of goods are produced, but the required payments are $d_0' = d_1' = \hat{d}$. It is easy to check that this offer satisfies all the constraints from (15) to (18), but makes the genuine money holder better off because he can choose $\hat{m}' = \hat{d} < \hat{m}$. That is the genuine money holder can save on the cost of carrying money, a contradiction. $\square$

A.3 Proof of Lemma 3

Proof. Suppose $\theta > 1$ in equilibrium for any $\{q_0, q_1, d\}$. This implies that $\alpha^b(\theta) < 1$ and $\alpha^s(\theta) = 1$. Consider an arbitrary $\theta' \in (1, \theta)$ such that $\alpha^b(\theta') > \alpha^b(\theta)$. Fix the same payment d, we can find a pair (q_0', q_1') together with θ' to satisfy constraints (15) to (18). Specifically, this can be done by choosing q_0' and q_1' to satisfy

$$
\begin{aligned}
\overline{S}^c &= \alpha^b(\theta')(1-\pi) u(q_0') - g, \\
0 &= P\pi q_1' + (1-\pi) q_0' + P\phi(\beta/\gamma)d - L.
\end{aligned}
$$

Thus we can find q_0' and q_1' such that $q_0 > q_0'$ and $q_1 < q_1'$. By doing so, the genuine money holder can gain since $\alpha^b(\theta')\pi u(q_1') > \alpha^b(\theta)\pi u(q_1)$ and all other terms remain the same in (14). Because θ' is arbitrary, this contradicts that the claim that $\theta > 1$ is optimal.

Next suppose $\theta < 1$ in equilibrium which implies that $\alpha^b(\theta) = 1$ and $\alpha^s(\theta) < 1$. Similar argument can be applied: for given arbitrary $\theta' \in (\theta, 1)$, we can find $d' < d$ and (q_0, q_1) are fixed so that all the constraints hold but the genuine money holder is strictly better off, a contradiction. $\square$

A.4 Proof of Proposition 1

Proof. Using Lemmas 2 and 3, and substituting (18) into (14), the maximization problem can be rewritten as

$$\max_{q_0,q_1}\left\{\pi u\left(q_1\right)-\frac{\gamma}{\beta P}q_1+\left(1-\pi\right)\left[u\left(q_0\right)-\frac{\gamma}{\beta P}q_0\right]\right\}$$
$$s.t.\ \overline{S}^c=\left(1-\pi\right)u\left(q_0\right)-g.$$

Let $q^{**}=\arg\max_q\left\{u\left(q\right)-(\gamma/\beta P)q\right\}$ for any given $P\in(0,1)$. q^{**} must satisfy

$$u'\left(q^{**}\right)=\frac{\gamma}{\beta P}>\frac{\gamma}{\beta}=u'\left(q_1\right).$$

By the concavity of u, it must be true that $q_1>q,^{**}$ and $u\left(q_1\right)-(\gamma/\beta)q_1>u\left(q^{**}\right)-(\gamma/\beta P)q^{**}$.

Suppose on the contrary that $q_0>q^{**}$ for any given $P\in(0,1)$ in equilibrium. We will show that if a seller deviates from (q_0,q_1) by offering (q_0',q_1') such that $q_0'=q_0-\varepsilon$ and $q_1'=q_1$, for any arbitrary small $\varepsilon>0$, he will attract all the genuine money holders by making the buyers using genuine money better off and those using counterfeits worst off. In this case, $P=1$ in his submarket and this leads to a contradiction. Obviously $q_0'<q_0$ results in a lower counterfeiter's surplus. Next, consider the effect of such a deviation on buyers holding genuine money. Since $q_0>q_0'>q^{**}$, it implies that

$$u\left(q_0\right)-\frac{\gamma}{\beta P}q_0<u\left(q_0'\right)-\frac{\gamma}{\beta P}q_0'<u\left(q^{**}\right)-\frac{\gamma}{\beta P}q^{**}<u\left(q_1\right)-\frac{\gamma}{\beta}q_1.$$

The inequalities hold because of the concavity of u as well. See Figure4 for illustration. Then we have the following results for the surplus of genuine money holders.

$$\begin{aligned}\overline{S}^g &= \pi\left(u\left(q_1\right)-\frac{\gamma}{\beta}q_1\right)+\left(1-\pi\right)\left(u\left(q_0\right)-\frac{\gamma}{\beta P}q_0\right),\\ &< \pi\left(u\left(q_1'\right)-\frac{\gamma}{\beta}q_1'\right)+\left(1-\pi\right)\left(u\left(q_0'\right)-\frac{\gamma}{\beta P}q_0'\right),\\ &= \overline{S}^{g\prime}.\end{aligned}$$

Therefore, the offer (q_0',q_1') will make the holder of genuine money better off and the holder of counterfeits worse off which contradicts (q_0,q_1) being an equilibrium.

$$\begin{aligned}\overline{S}^g &= \pi\left(u\left(q_1\right)-\frac{\gamma}{\beta}q_1\right)+\left(1-\pi\right)\left(u\left(q_0\right)-\frac{\gamma}{\beta P}q_0\right),\\ &< \pi\left(u\left(q_1'\right)-\frac{\gamma}{\beta}q_1'\right)+\left(1-\pi\right)\left(u\left(q_0'\right)-\frac{\gamma}{\beta P}q_0'\right),\\ &= \overline{S}^{g\prime}.\end{aligned}$$

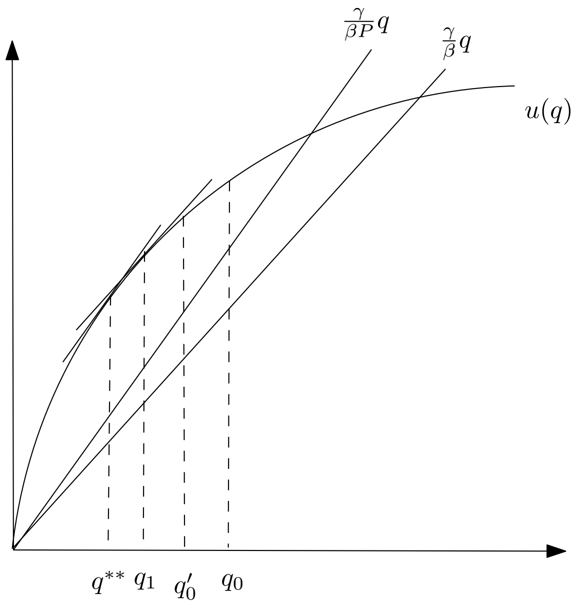

Figure 4: Properties of function u

Therefore, the offer (q_0', q_1') will make the holder of genuine money better off and the holder of counterfeits worse off which contradicts (q_0, q_1) being an equilibrium. □

A.5 Proof of Proposition 2

Proof. The decision on counterfeiting requires that the monetary equilibrium with counterfeiting exists if and only if $S^g = S^c$. Thus we have

$$P\pi\left(u\left(q_1\right) - \frac{\gamma}{\beta}q_1\right) + Pg - (1-\pi)\frac{\gamma}{\beta}q_0 = 0. \tag{35}$$

The first order condition pins down q_1 by $u'\left(q_1\right) = \gamma/\beta$. Next we will show that there exists a pair (P, q_0) satisfying equation (35). Denote the LHS of (35) by $\Delta\left(P, q_0\right)$. From Proposition 1, we know that $q_0 \in \left[0, q^{**}\left(P\right)\right]$ in equilibrium. When $q_0 = 0$, $\Delta\left(P, q_0\right) > 0$ for any $P \in (0, 1)$. It suffices to show that there exists a $P \in (0, 1)$ such that $\Delta\left(P, q^{**}\left(P\right)\right) < 0$. Then we can conclude from the mean value theorem that there is a $q_0 \in \left[0, q^{**}\left(P\right)\right]$ such that $\Delta\left(P, q_0\right) = 0$.

When $P = 0$, it implies that $q^{**}\left(P\right) = 0$, and$\Delta(P, q^{**}\left(P\right)) = \Delta\left(0, 0\right) = 0$.

When $P = 1$, $q^{**}\left(P\right) = q_1$, by condition (19) it implies

$$\Delta(P, q^{**}\left(P\right)) = \Delta\left(1, q_1\right) = \pi u\left(q_1\right) - \frac{\gamma}{\beta}q_1 + g < 0.$$

Since Δ is continuous, there must exist a $P \in (0, 1)$ such that $\Delta\left(P, q^{**}\left(P\right)\right) < 0$. □

A.6 Proof of Lemma 4

Proof. Suppose by contradiction that there exists an active submarket with $a = n$. This implies that uninformed sellers produce for genuine money. But then, buyers can gain by paying with counterfeits of the lowest quality because they know for sure that sellers cannot verify, a contradiction. $\square$

A.7 Proof of Proposition 3

Proof. Using the fact that $d_0 = d_1 = \hat{m}$ and $\alpha^b = \alpha^s = 1$, the problem (14) to (18), can be rewritten as

$$\max_{q_0,q_1} \left\{ \pi(H) u(q_1) + (1-\pi(H)) u(q_0) - \frac{\gamma}{\beta P} [L + P\pi(H) q_1 + (1-\pi(H))q_0] \right\} \tag{36}$$

$$s.t.\, \overline{S}^c = (1-\pi(H)) u(q_0) - g(H). \tag{37}$$

The objective function is obtained by substituting (18) into (14) and eliminating the payment $\hat{m}$. The first order conditions for q_1 and q_0 are

$$u'(q_1) = \frac{\gamma}{\beta}, \tag{38}$$

$$u'(q_0) = \frac{\gamma}{\beta P(1+\lambda)}, \tag{39}$$

where λ is the Lagrange multiplier of (37).

Suppose on the contrary that $q_0 > q_1$ for any given $P \in (0,1)$ in equilibrium. (38) and (39) implies that $1/[P(1+\lambda)] < 1 < 1/P$. We will show that if a seller deviates from (q_0, q_1) by offering (q_0', q_1') such that $q_0' = q_0 - \varepsilon$ and $q_1' = q_1$, for any arbitrary small $\varepsilon > 0$, he will attract all the genuine money holders by making the buyers using genuine money better off and those using counterfeits worst off. In this case, $P = 1$ in his submarket and this leads to a contradiction.

First, we consider the effect of such a deviation by the seller on counterfeiters. From (25) since H is a function of q_0 and $dH/dq_0 > 0$, $H(q_0') < H(q_0)$. Hence $\pi(H(q_0')) > \pi(H(q_0))$. Thus this results in a lower counterfeiter's surplus.

Second, consider the effect of such a deviation on buyers holding genuine money. Let $q^{**} = \arg\max_q \{u(q) - (\gamma/\beta P)q\}$ for any given $P \in (0,1)$. q^{**} must satisfy

$$u'(q^{**}) = \frac{\gamma}{\beta P} > \frac{\gamma}{\beta} = u'(q_1).$$

By the concavity of u, it must be true that $q_1 > q^{**}$ and $u(q_1) - (\gamma/\beta)q_1 > u(q^{**}) - (\gamma/\beta P)q^{**}$. Since $q_0 > q_0' > q_1$, it follows that $q_0 > q_0' > q_1 > q^{**}$ and thus

$$u(q_0) - \frac{\gamma}{\beta P} q_0 < u(q_0') - \frac{\gamma}{\beta P} q_0' < u(q^{**}) - \frac{\gamma}{\beta P} q^{**} < u(q_1) - \frac{\gamma}{\beta} q_1.$$

The inequalities hold because of the concavity of u as well. See Figure 4 for illustration. Then we have the following results for the surplus of genuine money holders.

$$\begin{aligned}
\overline{S}^g &= \pi\left(u(q_1)-\frac{\gamma}{\beta}q_1\right)+(1-\pi)\left(u(q_0)-\frac{\gamma}{\beta P}q_0\right)-\frac{\gamma}{\beta P}L,\\
&< \pi\left(u(q_1')-\frac{\gamma}{\beta}q_1'\right)+(1-\pi)\left(u(q_0')-\frac{\gamma}{\beta P}q_0'\right)-\frac{\gamma}{\beta P}L,\\
&< \pi'\left(u(q_1')-\frac{\gamma}{\beta}q_1'\right)+(1-\pi')\left(u(q_0')-\frac{\gamma}{\beta P}q_0'\right)-\frac{\gamma}{\beta P}L,\\
&= \overline{S}^{g\prime}.
\end{aligned}$$

Therefore, the offer (q_0', q_1') will make the holder of genuine money better off and the holder of counterfeits worse off which contradicts (q_0, q_1) being an equilibrium. $\square$

A.8 Proof of Proposition 4

Proof. A monetary equilibrium exists only if $P > 0$, i.e., some buyers would prefer to use genuine money. This requires that $S^g \geq S^c$. From (33) and (5), it means that

$$S^g - S^c = \pi(H)\left(u(q_1)-\frac{\gamma}{\beta}q_1\right)+g(H)-\frac{\gamma}{\beta P}(1-\pi(H))\,q_0-\frac{\gamma}{\beta P}L \geq 0, \quad (40)$$

where q_1 satisfies (32). Notice that from (25), H can be solved as a function of q_0, and more importantly $dH/dq_0 > 0$ and $H(0) = 0$. Thus, let $\Delta : \mathbb{R}^+ \to \mathbb{R}^+$ be a function of q_0 which represents the left hand side of (40):

$$\Delta(q_0) = \pi(H(q_0))\left(u(q_1)-\frac{\gamma}{\beta}q_1\right)+g(H(q_0))-\frac{\gamma}{\beta P}(1-\pi(H))\,q_0-\frac{\gamma}{\beta P}L.$$

Clearly Δ is differentiable. For any q_0,

$$\begin{aligned}
\frac{d\Delta}{dq_0} &= \pi' H'\left(u(q_1)-\frac{\gamma}{\beta}q_1\right)+g'H'-\frac{\gamma}{\beta P}(1-\pi)+\frac{\gamma}{\beta P}\pi' H' q_0,\\
&= \pi' H'\left[\left(u(q_1)-\frac{\gamma}{\beta}q_1\right)-\left(u(q_0)-\frac{\gamma}{\beta P}q_0\right)\right]-\frac{\gamma}{\beta P}(1-\pi).
\end{aligned}$$

The second equality uses the fact that $g' = -u(q_0)\,\pi'$ from (25). Since q_1 is the maximum of $u(q)-\gamma/\beta q$ and $P \leq 1$, $(u(q_1)-\gamma/\beta q_1) > (u(q_0)-\gamma/\beta q_0) \geq \left(u(q_0)-\frac{\gamma}{\beta P}q_0\right)$. Because $\pi' < 0$ and $H' > 0$, we conclude that $d\Delta/dq_0 < 0$.

Sufficiency. If $q_0 = 0$, then $S^c = 0$. Suppose $\Delta(0) = \pi(0)(u(q_1)-\gamma/\beta q_1) - \gamma/\beta L \geq 0$, then $S^g \geq 0$ and thus $P = 1$. By the continuity of Δ it is possible to find a q_0 in the open neighborhood of 0 such that $\Delta(q_0) \geq 0$.

Necessity. Obviously, if $\Delta(0) < 0$, then for any q_0, $\Delta(q_0) < \Delta(0) < 0$ which violates condition (40).

The sufficient and necessary conditions for the existence of a monetary equilibrium with counterfeiting is $S^g = S^c$ which implies

$$P\pi(H)\left(u(q_1) - \frac{\gamma}{\beta}q_1\right) + Pg(H) - \frac{\gamma}{\beta}(1-\pi(H))\,q_0 = \frac{\gamma}{\beta}L, \tag{41}$$

for some $P \in (0,1)$. Denote left hand side of (41) by $\tilde{\Delta}(q_0, P)$. Taking the derivative of $\tilde{\Delta}$ with respect to q_0 and P, we can see that$\tilde{\Delta}$ is strictly decreasing in q_0 and strictly increasing in P, hence $\tilde{\Delta}(0,1) = \max_{(q_0,P)} \tilde{\Delta}(q_0, P)$. Notice that $\tilde{\Delta}(q_0, 0) < \gamma/\beta L$ for any $q_0 > 0$, if $\tilde{\Delta}(0,1) > \gamma/\beta L$, then by the mean value theorem, we can find (q_0, P) in the open neighborhood of $(0,1)$ such that (41) holds and equilibrium exists. When $\tilde{\Delta}(0,1) \leq \gamma/\beta L$, no monetary equilibrium with counterfeiting exists because $\tilde{\Delta}(q_0, P) < \tilde{\Delta}(0,1) \leq \gamma/\beta L$ for all $P < 1$. □

A.9 Proof of Results 1 and 2

Proof. The equilibrium conditions for the monetary equilibrium with counterfeiting are summarized by equations (41), (32) and (25). For given q_0, rewrite (41) as

$$P\left[\pi(H(q_0))\left(u(q_1) - \frac{\gamma}{\beta}q_1\right) + g(H(q_0))\right] = \frac{\gamma}{\beta}\left[L + (1-\pi(H(q_0)))\,q_0\right]. \tag{42}$$

Notice from (25), H is fixed for given q_0, so is π and g. If γ increases, the right hand side of (42) rises. By the concavity of u, the term inside the bracket of the left hand side of (42) decreases as $\gamma \uparrow$. Therefore to keep the equality of (42), P must go up. Similarly, if L goes up, P will rise as well. We establish that $dP/d\gamma > 0$ and $dP/dL > 0$.

For given P, totally differentiate (42), we have

$$\begin{aligned}\frac{dq_0}{d\gamma} &= \frac{P\pi q_1 + L + (1-\pi)\,q_0}{\beta\left\{\pi' h'\left[P\left(u(q_1) - \frac{\gamma}{\beta}q_1\right) - P\left(u(q_0) - \frac{\gamma}{\beta}q_0\right) + (1-P)\frac{\gamma}{\beta}q_0\right] - \frac{\gamma}{\beta}(1-\pi)\right\}} \\ &< 0\end{aligned}$$

and

$$\begin{aligned}\frac{dq_0}{dL} &= \frac{\gamma}{\beta\left\{\pi' h'\left[P\left(u(q_1) - \frac{\gamma}{\beta}q_1\right) - P\left(u(q_0) - \frac{\gamma}{\beta}q_0\right) + (1-P)\frac{\gamma}{\beta}q_0\right] - \frac{\gamma}{\beta}(1-\pi)\right\}} \\ &< 0\end{aligned}$$

□

A.10 Proof of Result 3

Proof. Rewrite equation (25) as $\delta g_0'(H) = -\pi'(H)\,u(q_0)$. For given H, q_0 is increasing in δ, while for given q_0, H is decreasing in δ. Therefore, $dP/d\delta > 0$ is derived immediately from (42) for any given H. However if q_0 is fixed, from (42) we obtain

$$\frac{dP}{dH} = -\frac{\pi\left(u(q_1) - \frac{\gamma}{\beta}q_1 + g\right)}{\pi'\left[P\left(u(q_1) - \frac{\gamma}{\beta}q_1\right) - P\left(u(q_0) - \frac{\gamma}{\beta}q_0\right) + (1-P)\frac{\gamma}{\beta}q_0\right]} > 0.$$

$$\frac{dP}{dH} = -\frac{\pi'\left[P\left(u(q_1) - \frac{\gamma}{\beta}q_1\right) - P\left(u(q_0) - \frac{\gamma}{\beta}q_0\right) + (1-P)\frac{\gamma}{\beta}q_0\right]}{\pi\left(u(q_1) - \frac{\gamma}{\beta}q_1\right) + g} > 0.$$

Hence $dP/d\delta < 0$. □

A.11 Proof of Proposition 5

Proof. The pooling outcome must satisfy the seller's incentive compatibility constraint (33). Since $L \geq 0$, it requires that $q_0 \geq Pq_1$ for given P. $q_0 \leq q_1$ follows from Proposition 1.

The pooling equilibrium exists if and only if $S^g \geq S^c$ and the seller has no incentive to break the pooling offer. This implies

$$\begin{aligned} P\pi\left(u(q_1) - \frac{\gamma}{\beta}q_1\right) + Pg - \frac{\gamma}{\beta}(1-\pi)\,q_0 &\geq \frac{\gamma}{\beta}L, &(43)\\ \pi(q_0 - Pq_1) &= L, &(44)\end{aligned}$$

where q_1 satisfies (32). $P < 1$ and the first inequality becomes "=" if $S^g = S^c$. Replace q_0 in (43) using (44), we have

$$\pi P\left(\pi u(q_1) - \frac{\gamma}{\beta}q_1 + g\right) \geq \frac{\gamma}{\beta}L. \tag{45}$$

The first order condition on H suggests

$$u\left(\frac{L}{\pi} + Pq_1\right)\pi' + g' = 0. \tag{46}$$

Therefore, (45) and (46) consist of a sufficient and necessary condition for the existence of pooling equilibrium. We want to show that conditions in set B at (34) satisfies (45) and (46). Obviously, if $(\gamma, L) \in B$, the above condition is automatically satisfied by setting $P = 1$, i.e., monetary equilibrium without counterfeiting. For the case $P < 1$, it requires that

$$\pi P\left(\pi u(q_1) - \frac{\gamma}{\beta}q_1 + g\right) = \frac{\gamma}{\beta}L. \tag{47}$$

Because from (46), H can be written as a function of P, the left hand side of (47) can be written as a function of P as well. Denote it as $\Delta(P)$. We want to prove that $\Delta(P) = \gamma/\beta L$ has a solution for some $P \in (0,1)$. Since $\Delta(0) < \gamma/\beta L$ and $\Delta(1) > \gamma/\beta L$ if $(\gamma, L) \in int(B)$, using the mean value theorem, there is a $P \in (0,1)$ such that Equation (47) holds. □

A.12 Proof of Result 4

Proof. The equilibrium conditions for a pooling outcome can be written as

$$\begin{aligned} P\left(\pi(H)u(q_1) - \frac{\gamma}{\beta}q_1 + g(H)\right) - \frac{\gamma}{\beta}\frac{L}{\pi(H)} &= 0, & (48)\\ u'(q_1) - \frac{\gamma}{\beta} &= 0, & (49)\\ \frac{\pi'(H)}{g'(H)} + \frac{1}{u\left(\frac{L}{\pi(H)} + Pq_1\right)} &= 0, & (50) \end{aligned}$$

by collecting (32), (43) and (46). Let $F(P, q_1, H)$ be the left hand side of the above equation system. γ and L are the parameters. The derivative matrix of F with respect to (γ, L) is

$$D_{(\gamma,L,\delta)}F = \begin{bmatrix} -\frac{P}{\beta}q_1 - \frac{L}{\beta\pi} & -\frac{\gamma}{\beta\pi} & g_0 \\ -\frac{1}{\beta} & 0 & 0 \\ 0 & -\frac{u'(q_0)}{u^2(q_0)\pi} & -\frac{\pi'}{\delta^2 g_0'} \end{bmatrix},$$

where $q_0 = L/\pi + Pq_1$, and $g_0 = g/\delta$. While the Jacobian matrix of the system with respect to (P, q_1, H) is

$$D_{(P,q_1,H)}F = \begin{bmatrix} \pi u(q_1) - \frac{\gamma}{\beta}q_1 + g & P\left(\pi u'(q_1) - \frac{\gamma}{\beta}\right) & P(\pi' u(q_1) + g') + \frac{\gamma L}{\beta\pi^2}\pi' \\ 0 & u''(q_1) & 0 \\ -\frac{u'(q_0)}{u^2(q_0)}q_1 & -\frac{u'(q_0)}{u^2(q_0)}P & \frac{u'(q_0)\pi' L}{u^2(q_0)\pi^2} - \frac{\pi'' g' - \pi' g''}{g'^2} \end{bmatrix}.$$

The determinant of $D_{(P,q_1,H)}F$ is

$$det\left(D_{(P,q_1,H)}F\right) = u''(q_1)\left[\begin{array}{l} \left(\pi u(q_1) - \frac{\gamma}{\beta}q_1 + g\right)\left(\frac{u'(q_0)\pi' L}{u^2(q_0)\pi^2} - \frac{\pi'' g' - \pi' g''}{g'^2}\right) \\ + \left[P(\pi' u(q_1) + g') + \frac{\gamma L}{\beta\pi^2}\pi'\right]\frac{u'(q_0)}{u^2(q_0)}q_1 \end{array}\right].$$

From (48), $\pi u(q_1) - \frac{\gamma}{\beta}q_1 + g > 0$. And from (50),

$$P(\pi' u(q_1) + g') = P\pi'(u(q_1) - u(q_0)) < 0$$

since $q_1 > q_0$ in equilibrium. Together with $\pi' < 0$, $\pi'' > 0$, $g' > 0$, $g'' > 0$, $u' > 0$ and $u'' < 0$, we can see that $det\left(D_{(P,q_1,H)}F\right) > 0$. Apply the implicit function theorem, we have

$$\frac{dP}{d\gamma} = \frac{-1}{det\left(D_{(P,q_1,H)}F\right)} \left\{ \begin{array}{l} \left[\left(-\frac{Pq_1}{\beta} - \frac{L}{\beta\pi}\right) u''(q_1) + \frac{P}{\beta}\left(\pi u'(q_1) - \frac{\gamma}{\beta}\right)\right] \\ \times \left(\frac{u'(q_0)\pi' L}{u^2(q_0)\pi^2} - \frac{\pi'' g' - \pi' g''}{g'^2}\right) \\ + \frac{Pu'(q_0)}{\beta u^2(q_0)}\left[P\left(\pi' u(q_1) + g'\right) + \frac{\gamma L}{\beta\pi^2}\pi'\right] \end{array} \right\}.$$

The first term in the bracket is negative because

$$\begin{aligned} &\left(-\frac{Pq_1}{\beta} - \frac{L}{\beta\pi}\right) u''(q_1) + \frac{P}{\beta}\left(\pi u'(q_1) - \frac{\gamma}{\beta}\right) \\ &= \left(-\frac{Pq_1}{\beta} - \frac{L}{\beta\pi}\right) u''(q_1) + \frac{P}{\beta}\left(\pi u'(q_1) - u'(q_1)\right) \\ &= \frac{P}{\beta}\left[-q_1 u''(q_1) - (1-\pi) u'(q_1)\right] - \frac{L}{\beta\pi} u''(q_1) \\ &> 0 \end{aligned}$$

The last inequality follows by the assumption 1:

$$-q_1 u''(q_1)/u'(q_1) > 1 - \pi^* > 1 - \pi.$$

The second term in the bracket is also negative. Then, $dP/d\gamma > 0$. Similarly,

$$\begin{aligned} \frac{dP}{dL} &= \frac{1}{det\left(D_{(P,q_1,H)}F\right)} \left\{ \begin{array}{l} \frac{\gamma}{\beta\pi} u''(q_1)\left(\frac{u'(q_0)\pi' L}{u^2(q_0)\pi^2} - \frac{\pi'' g' - \pi' g''}{g'^2}\right) \\ + \frac{u'(q_0)u''(q_1)}{u^2(q_0)\pi}\left[P\left(\pi' u(q_1) + g'\right) + \frac{\gamma L}{\beta\pi^2}\pi'\right] \end{array} \right\}, \\ &> 0. \end{aligned}$$

$$\begin{aligned} \frac{dP}{d\delta} &= \frac{-1}{det\left(D_{(P,q_1,H)}F\right)} \left\{ \begin{array}{l} g_0 u''(q_1)\left(\frac{u'(q_0)\pi' L}{u^2(q_0)\pi^2} - \frac{\pi'' g' - \pi' g''}{g'^2}\right) \\ - \frac{\pi' u''(q_1)}{\delta^2 g_0'}\left[P\left(\pi' u(q_1) + g'\right) + \frac{\gamma L}{\beta\pi^2}\pi'\right] \end{array} \right\} \\ &< 0 \end{aligned}$$

$$\begin{aligned} \frac{dH}{d\gamma} &= \frac{-1}{det\left(D_{(P,q_1,H)}F\right)} \left\{ \begin{array}{l} \left[\frac{u'(q_0)q_1}{u^2(q_0)}\left(-\frac{Pq_1}{\beta} - \frac{L}{\beta\pi}\right) u''(q_1) + \frac{P}{\beta}\left(\pi u'(q_1) - \frac{\gamma}{\beta}\right)\right] \\ + \frac{u'(q_0)P}{u^2(q_0)}\left(\pi u(q_1) - \frac{\gamma}{\beta}q_1 + g\right) \end{array} \right\}, \\ &< 0 \end{aligned}$$

and

$$\frac{dH}{dL} = \frac{1}{det\left(D_{(P,q_1,H)}F\right)} \left\{ \begin{array}{l} \dfrac{\gamma u''\left(q_1\right) u'\left(q_0\right) q_1}{\beta \pi u^2\left(q_0\right)} \\ + \dfrac{u'\left(q_0\right) u''\left(q_1\right)}{\pi u^2\left(q_0\right)} \left(\pi u\left(q_1\right) - \dfrac{\gamma}{\beta} q_1 + g\right) \end{array} \right\} < 0$$

$$\frac{dH}{d\delta} = \frac{-1}{det\left(D_{(P,q_1,H)}F\right)} \left\{ \begin{array}{l} \dfrac{g_0 u''\left(q_1\right) u'\left(q_0\right) q_1}{\beta \pi u^2\left(q_0\right)} \\ + - \dfrac{\pi' u''\left(q_1\right)}{\delta^2 g_0'} \left(\pi u\left(q_1\right) - \dfrac{\gamma}{\beta} q_1 + g\right) \end{array} \right\} > 0$$

$\square$

References

"Bank of Canada Annual Report," (2008).

Cavalcanti, R. and E. Nosal, "Couterfeiting as Private Money in Mechanism Design," Federal Reserve Bank of Cleveland Working Paper 07-16, 2007.

Green, E. J. and W. E. Weber, "Will the New 100 Dollar Bill Decrease Counterfeiting," *Federal Reserve Bank of Minneapolis Quaterly Review* 20 (1996), 3–10.

Guerrieri, V., R. Shimer and R. Wright, "Adverse Selection in Competitive Search Equilibrium," *Econometrica* 78 (November 2010), 1823–1862.

Kiyotaki, N. and R. Wright, "A Search-Theoretic Approach to Monetary Economics," *American Economic Review* 83 (1993), 63–77.

Kultti, K., "A Monetary Economy with Couterfeiting," *Journal of Economics* 63 (1996), 175–186.

Lagos, R. and R. Wright, "A Unified Framework for Monetary Theory and Policy Analysis," *Journal of Political Economy* 113 (2005), 463–484.

Lengwiler, Y., "A Model of Money Counterfeits," *Journal of Economics* 65 (1997), 123–132.

Li, Y. and G. Rocheteau, "On the Threat of Counterfeiting," Federal Reserve Bank of Cleveland Working Paper 08-09, November 2008.

Moen, E. R., "Competitive Search Equilibrium," *Journal of Political Economy* 105 (1997), 385–411.

Monnet, C., "Counterfeiting and Inflation," European Central Bank Working Paper No. 512, August 2005.

Nosal, E. and N. Wallace, "A Model of (the threat of) counterfeiting," *Journal of Monetary Economics* 54 (2007), 994–1001.

Quercioli, E. and L. Smith, "Couterfeit Money," mimeo., 2007.

Rocheteau, G. and R. Wright, "Money in Search Equilibrium, in Competitive Equilibrium, and in Competitive Search Equilibrium," *Econometrica* 73 (2005), 175–202.

Williamson, S. D., "Private Money and Couterfeiting," *Federal Reserve Bank of Richmond Economic Quarterly* 88 (2002), 37–57.